Sustainable monetary policies

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Abstract

This paper uses a model with utility maximizing households, monopolistically competitive firms, and sticky goods prices to derive a version of Barro and Gordon’s time consistency problem for monetary policy where the government’s objectives are consistent with a representative household’s preferences. The paper applies the methods of Chari and Kehoe to characterize the entire set of time consistent, or sustainable, outcomes. It goes on to find conditions under which the Friedman rule, the optimal policy under commitment, can be supported when the government lacks a commitment technology.

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1. Introduction

Barro and Gordon (1983), building on an example first presented by Kydland and Prescott (1977), develop what has now become the most widely-used model for studying the time consistency of monetary policy. Undoubtedly, much of the Barro-Gordon model’s popularity stems from its elegance and simplicity. For indeed, the model consists of just two equations: an expectational Phillips curve relationship that links aggregate output to surprise inflation and an objective function for the government that assigns benefits to higher output and costs to realized inflation. This simplicity, however, has also provided fodder for the model’s critics. In particular, surveys by Cukierman (1986) and Blackburn and Christensen (1989) both question whether the government’s objectives in the Barro-Gordon model can plausibly reflect the preferences of an optimizing private sector.
This paper responds to this question by deriving a version of the Barro–Gordon problem from a more detailed model with utility maximizing households, monopolistically competitive firms, and sticky goods prices. In this model, the government seeks to maximize a representative household’s utility function; government and private objectives coincide. The joint presence of monopolistically competitive firms and sticky goods prices implies that the government can increase welfare by creating surprise inflation, just as in Barro and Gordon’s original model. Here, however, expected inflation causes agents to inefficiently economize on their holdings of real cash balances and is therefore the source of welfare costs; Barro and Gordon, in contrast, assign costs to realized inflation. Thus, the more detailed model developed here provides a sharper view of the government’s objectives and their relation to those of the private sector than can be found in previous studies of time consistent monetary policy.

The model also provides a new setting in which the methods of Chari and Kehoe (1990) can be applied to characterize the entire set of time consistent, or sustainable, equilibria. These methods adapt Abreu’s (1988) optimal penal codes to policy games played between a benevolent government and a competitive private sector. Chari and Kehoe demonstrate that any sustainable outcome can be supported by a reputational equilibrium in which private expectations display an extreme form of trigger-like behavior: a single deviation by the government from its announced plan causes the economy to revert permanently to its worst possible outcome.1 Thus, identifying the worst possible outcome becomes a crucial step in the analysis. Examples presented by Rogoff (1989) and Stokey (1991) suggest that the worst outcome may be quite complicated in the original Barro–Gordon model. Here, however, the government’s objectives differ from those assumed by Barro and Gordon, making the worst possible outcome relatively easy to describe.

The results show that when the government can commit to a policy at the beginning of time, it chooses to follow the Friedman (1969) rule, steadily contracting the money supply so that the nominal interest rate equals zero. This optimal policy under commitment can sometimes be supported in a reputational equilibrium when the government lacks a commitment technology. The specific conditions that must be satisfied depend on the parameters of the representative household’s utility function. It is possible, therefore, to identify reasonable magnitudes for these parameters and ask whether the conditions allowing the Friedman rule to be supported without commitment are likely to hold. Other versions of the Barro–Gordon model, in contrast, do not explicitly tie the government’s objectives to the preferences of private agents; these models provide little guidance for assigning values to the key parameters.

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1 Stokey (1991) derives similar results. The notation, definitions, and arguments used here, however, follow those of Chari and Kehoe (1990) more closely. Chari and Kehoe (1993a, b) apply these methods to study the government’s decision to default on its debt.
Before launching into the analysis, mention should be made of previous work by Cubitt (1993), who also derives a version of the Barro–Gordon problem from a model with utility maximizing households, monopolistically competitive firms, and sticky goods prices. Cubitt's model, however, is a static one; it cannot be used to consider reputational equilibria. The present study can therefore be classified as an extension of Cubitt's, where the setting becomes fully dynamic.

2. The economic environment

The model takes most of its structure from those developed by Svensson (1986), Rotemberg (1987), and Blanchard and Fischer (1989, Chapter 8). Households face cash-in-advance constraints on their consumption purchases. These constraints give rise to an interest-elastic demand for real balances; expected inflation causes agents to inefficiently economize on their money holdings. Firms operate in monopolistically competitive markets and are required to set prices for their output one period in advance. Monopolistic competition implies that equilibrium output falls short of the efficient level; sticky prices allow unanticipated money to have real effects. Thus, the government faces a trade-off between the costs of expected inflation and the benefits of unexpected inflation. This trade-off is also present in Grossman's (1990) version of the Barro–Gordon (1983) model. It differs, however, from the trade-off originally considered by Barro and Gordon, who assign costs to realized, rather than expected, inflation.

The economy consists of a representative household, a continuum of firms indexed by \( i \in [0, 1] \), and a government. Each firm produces a distinct, perishable consumption good; hence goods in the economy are also indexed by \( i \in [0, 1] \), where firm \( i \) produces good \( i \). Preferences and technologies display enough symmetry, however, to allow the analysis to consider the activities of a representative firm, identified by the generic index \( i \).

The government controls the money supply by making a lump-sum transfer \((x_t - 1)M_t^s\) to the representative household at each date \( t = 0, 1, 2, \ldots \). Thus, the per-household money stock \( M_t^s \) at the beginning of time \( t = 0, 1, 2, \ldots \) obeys

\[
M_{t+1}^s = x_t M_t^s.
\]

A choice of nominal units yields the normalization \( M_0^s = 1 \).

The representative household trades bonds as well as money. Bonds costing the household \( B_{t+1}/R_t \) dollars at time \( t \) return \( B_{t+1} \) dollars at time \( t + 1 \), where \( R_t \) is the gross nominal interest rate between \( t \) and \( t + 1 \). Bonds are available in zero net supply, so \( B_{t+1} = 0 \) must hold in equilibrium for all \( t = 0, 1, 2, \ldots \).

During each period \( t = 0, 1, 2, \ldots \), events unfold as follows. The representative household enters time \( t \) with money \( M_t \) and bonds \( B_t \). The representative firm enters time \( t \) having fixed a nominal price \( P_t(i) \) for its output.
At the beginning of time \( t \), the representative household receives the nominal transfer \((x_t - 1)M_t^*\). Next, the household's bonds \( B_t \) mature, bringing its money holdings to \( M_t + (x_t - 1)M_t^* + B_t \). The household uses some of this cash to purchase new bonds of value \( B_{t+1}/R_t \) and carries the rest into the goods market.

The description of goods production and trade draws on Lucas' (1980) interpretation of the cash-in-advance model. The representative household consists of two members: a shopper and a worker. During time \( t \), the shopper purchases \( c_t(i) \) units of each good \( i \) from firm \( i \) at its fixed price \( P_t(i) \), subject to the cash-in-advance constraint

\[
\int_0^1 P_t(i)c_t(i)di \leq M_t + (x_t - 1)M_t^* + B_t - B_{t+1}/R_t.
\]

The worker, meanwhile, supplies \( n_t(i) \) units of labor to each firm \( i \). He receives the nominal wage \( W_t \).

The household's preferences are described by the utility function

\[
\sum_{i=0}^{\infty} \beta^i (c_t^\alpha/n_t),
\]

where \( 0 < \beta < 1, \ 0 < \alpha < 1 \), and the composite goods \( c_t \) and \( n_t \) are defined by

\[
c_t = \left[ \int_0^1 c_t(i)^{(\theta-1)/\theta} di \right]^{\theta/(\theta-1)},
\]

where \( 1 < \theta \), and

\[
n_t = \int_0^1 n_t(i) di.
\]

The restrictions \( 0 < \alpha < 1 \) ensure that the household's utility is concave in \( c_t \) but still well-defined when \( c_t = 0 \). The restriction \( 1 < \theta \) is required for the existence of equilibria with monopolistically competitive firms. Part 1 of the appendix shows that with these preferences, the household's demand for good \( i \) can be written

\[
c_t(i) = [P_t(i)/P_t]^{-\theta}(x_tM_t^*/P_t),
\]

where the aggregate price level \( P_t \) is defined by

\[
P_t = \left[ \int_0^1 P_t(i)^{1-\theta} di \right]^{1/(1-\theta)}.
\]

The representative firm must sell output on demand at its fixed price \( P_t(i) \) during time \( t \). It produces this output according to a linear technology that yields
one unit of good $i$ for every unit of labor input. At the end of time $t$, the firm makes its wage payment and pays out any profit as a dividend to the representative household. In light of the linear production technology, this dividend payment $D_t(i)$ equals price minus wage times quantity sold. Thus, using the demand function (3),

$$D_t(i) = [P_t(i) - W_t][P_t(i)/P_t]^{-\theta}(x_t M_t^i/P_t).$$

The firm then sets next period’s price $P_{t+1}(i)$ to maximize next period’s dividend $D_{t+1}(i)$.

The representative household uses its unspent cash and its dividend and wage receipts as sources of funds with which to accumulate the money $M_{t+1}$ that it carries into time $t + 1$. It faces the budget constraint

$$\int_0^1 P_t(i)c_t(i) di + B_{t+1}/R_t + M_{t+1} \leq M_t + (x_t - 1)M_t^i + B_t + \int_0^1 D_t(i) di + W_tm_t.$$ 

As a first step in characterizing an equilibrium for this economy, define $m_t = M_t/M_t^i$, $b_t = B_t/M_t^i$, $d_t(i) = D_t(i)/M_t^i$, $w_t = W_t/M_t^i$, $p_t(i) = P_t(i)/M_t^i$, and $p_t = P_t/M_t^i$. In terms of these scaled nominal variables, the representative household’s budget and cash-in-advance constraints become

$$\int_0^1 p_t(i)c_t(i) di + b_{t+1}x_t/R_t + m_{t+1}x_t \leq m_t + (x_t - 1) + b_t + \int_0^1 d_t(i) di + w_tm_t \quad (4)$$

and

$$\int_0^1 p_t(i)c_t(i) di \leq m_t + (x_t - 1) + b_t - b_{t+1}x_t/R_t, \quad (5)$$

while the representative firm’s dividend payment becomes

$$d_t(i) = [p_t(i) - w_t][p_t(i)/p_t]^{-\theta}(x_t/p_t). \quad (6)$$

3. Monetary policy with commitment

Under commitment, the government chooses a policy at the beginning of time, before firms set their prices for $t = 0$. Formally, a policy for the government is a sequence $x = \{x_t\}_{t=0}^\infty$ of money growth rates, where $x_t \in [\beta, \bar{x}]$ for all $t = 0, 1, 2, \ldots$. These bounds on money growth ensure the existence of a monetary equilibrium. The lower bound helps guarantee that the net nominal
interest rate \( R_t - 1 \) is nonnegative. The upper bound \( \bar{x} < \infty \) is introduced, following Calvo (1978), to guarantee that private agents never abandon the use of money altogether.

Firm and household behavior, given a policy \( x \), is summarized by allocation rules \( \pi^i(x) \), \( i \in [0,1] \), and \( h(x) \). The representative firm's rule \( \pi^i(x) \) specifies settings for \( p_t(i), t = 0,1,2, \ldots, \) for each possible \( x \). The representative household's rule \( h(x) \) specifies choices for \( H_t = (c_t, c_t(i), n_t, m_{t+1}, b_{t+1}), t = 0,1,2, \ldots, \) for each possible \( x \). Let \( \pi \) refer to the set of functions \( \pi^i \) for all \( i \in [0,1] \). Then \( \pi \) and \( h \) map policies \( x \) into allocations \( (\Pi, H) \), where \( \Pi = \{ \Pi^i | i \in [0,1] \}, \Pi^i = \{ p_t(i) | t = 0,1,2, \ldots, \} \), and \( H = \{ H_t | t = 0,1,2, \ldots \} \). Thus, the allocation \( (\Pi, H) \) describes the sequence of equilibrium prices and quantities that obtains when the government adopts the policy \( x \) and private agents respond according to \( \pi \) and \( h \).

Given \( x \), \( \pi^j(x) \) for all \( j \in [0,1], j \neq i \), and \( h(x) \), the representative firm's choice of \( \pi^i(x) \) solves

(FO) Maximize (6) for each \( t = 0,1,2, \ldots, \) taking \( w_t \) and \( p_t \) as given for all \( t = 0,1,2, \ldots. \)

Given \( x \) and \( \pi^i(x) \) for all \( i \in [0,1] \), the representative household's choice of \( h(x) \) solves

(HO) Maximize (1) subject to (2), (4), and (5) for each \( t = 0,1,2, \ldots, \) taking \( d_t(i), w_t, \) and \( R_t \) as given for all \( i \in [0,1] \) and \( t = 0,1,2, \ldots. \)

In addition, \( h(x) \) must be consistent with the market clearing conditions

(MO) \( m_{t+1} = 1, b_{t+1} = 0, \) and

\[
\int_0^1 c_t(i) \, di = n_t
\]

for all \( t = 0,1,2, \ldots. \)

Following Chari and Kehoe (1990), optimal policy under commitment is defined as part of a Ramsey equilibrium. A Ramsey equilibrium consists of a policy \( x \) and a set of allocation rules \( (\pi, h) \) that satisfy: (i) for every \( x \), each \( \pi^i(x) \) solves (FO) given \( \pi^j(x) \) for all \( j \in [0,1], j \neq i \), and \( h(x) \); (ii) for every \( x \), \( h(x) \) solves (HO) given \( \pi^i(x) \) for all \( i \in [0,1] \); (iii) for every \( x \), \( h(x) \) is consistent with (MO); (iv) \( x \) maximizes (1) when \( c_t \) and \( n_t \) are determined by \( \pi^i(x) \) for all \( i \in [0,1] \) and \( h(x) \). Given the Ramsey equilibrium \( (x, \pi, h) \), the corresponding Ramsey outcome \( (x, \Pi, H) \) is defined by \( \Pi^i = \pi^i(x) \) for all \( i \in [0,1] \) and \( H = h(x) \). Thus, the Ramsey outcome describes the sequence of equilibrium prices and quantities that obtains when the government adopts the optimal monetary policy and private agents respond by maximizing utility and profits.
To begin characterizing this economy's Ramsey outcome, define an *equilibrium under commitment* as a policy $x$ and an allocation $(\Pi, H)$ that satisfy: (i) given $x$, $\Pi^j$ for all $j \in [0, 1], j \neq i$, and $H$, each $\Pi^i$ solves (FO); (ii) given $x$ and $\Pi^i$ for all $i \in [0, 1], H$ solves (HO); (iii) $H$ is consistent with (MO). The definition of an equilibrium under commitment parallels that of a Ramsey outcome, but drops the requirement that $x$ be chosen optimally. Thus, this definition allows the Ramsey outcome to be constructed by first characterizing the entire set of equilibria under commitment and then finding the member of this set that maximizes the household's utility function (1).

Accordingly, fix $x = \{x_t | t = 0, 1, 2, \ldots\}$ and consider the household's problem (HO). Part 1 of the appendix shows that necessary and sufficient conditions for a solution for this problem reduce to

$$c_t = n_t = x_t/p_t,$$  

$$w_t = (1/\beta)x_t x_{t+1}(p_{t+1}/x_{t+1})^\beta,$$  

and

$$\lim_{t \to \infty} \beta'(c_t^\beta/x_t) = 0$$  

when the market clearing conditions (MO) are imposed. Next, consider the firm’s problem (FO). Part 1 of the appendix also shows that the unique solution to this problem implies that all firms charge the same fixed markup of price over marginal cost, so that $p_t(i) = p_t$ for all $i \in [0, 1]$, where

$$p_t = \theta/(\theta - 1)w_t.$$  

Combining (7), (8), and (10) yields a difference equation for $c_t$ that must hold in any equilibrium under commitment:

$$c_t = \beta[(\theta - 1)/\theta](c_{t+1}^\beta/x_{t+1}).$$  

Part 1 of the appendix establishes

**Proposition 3.1.** The Ramsey outcome can be found by maximizing (1) subject to the constraints (9), (11), $c_t = n_t$, and $x_t \in [\beta, \bar{x}]$ for all $t = 0, 1, 2, \ldots$.

Part 2 of the appendix establishes

**Proposition 3.2.** The Ramsey outcome has $x_t = \beta$ and $R_t = 1$ for all $t = 0, 1, 2, \ldots$.

When the government commits to a policy at the beginning of time, it loses the ability to push equilibrium output closer to its efficient level by creating surprise
inflation. Thus, the source of the time consistency problem vanishes, and only the costly effects of anticipated inflation remain. To see this, note that under constant money growth, $x_t = \bar{x}$ for all $t = 0, 1, 2, \ldots$, the solution to (11) has $c_t = \bar{c}$ for all $t = 0, 1, 2, \ldots$, where

$$\bar{c} = \left[\beta(\theta - 1)/\theta \bar{x}\right]^{1/(1-\alpha)}.$$  

(12)

Faster money growth provides the representative household with an incentive to economize on its cash balances by purchasing fewer goods and enjoying more leisure. Hence, (12) shows that $\bar{c}$ is strictly decreasing in $\bar{x}$. The solution to the Ramsey problem eliminates this source of inefficiency by requiring policy to follow the Friedman (1969) rule: contract the money supply so that the net nominal interest rate equals zero. Eq. (12) also implies that even with $x_t = \beta$ for all $t = 0, 1, 2, \ldots$, consumption falls below its efficient level $c^* = 1$. The Ramsey policy cannot remove the other source of inefficiency in this economy: the monopolistically competitive market structure.

The next section turns to the case where the government can no longer commit to a policy at the beginning of time. In characterizing equilibria in this case, an additional result, also established in part 2 of the appendix, proves useful:

**Proposition 3.3.** The worst equilibrium under commitment has $x_t = \bar{x}$ for all $t = 0, 1, 2, \ldots$.

Eq. (12) implies that with $x_t = \bar{x}$ for all $t = 0, 1, 2, \ldots$, $c_t$ approaches zero as $\bar{x}$ becomes arbitrarily large. Thus, in the limit, the worst equilibrium under commitment is autarkic; high inflation drives out market activity altogether.

4. **Monetary policy without commitment**

When the government lacks a commitment technology, it must set policy sequentially, choosing $x_t$ at the beginning of each period $t$. Since the government decides on $x_t$ after firms have fixed their prices $p_t(i)$, and since monopolistic competition implies that output falls below the efficient level, the government is tempted in each period to set money growth unexpectedly high. In equilibrium, however, firms recognize that the government has this incentive and set their prices accordingly. The government’s attempts to increase output by creating positive monetary surprises lead only to higher rates of inflation. Thus, the monopolistically competitive market structure coupled with the rigidity in nominal goods prices gives rise to a version of Barro and Gordon’s (1983) time consistency problem for monetary policy in this model.
Without commitment, firms, households, and the government can revise their plans at each date \( t = 0, 1, 2, \ldots \), based on the history of government policy. In some equilibria, the government's policy choice today affects market expectations in the future; in these equilibria, the government appears to care about its reputation.

For each \( t = 0, 1, 2, \ldots \), denote the history of government policy through time \( t \) by \( \xi_t = \{x_s|s = 0, 1, \ldots, t\} \). Also, define \( \xi_{-1} = \emptyset \). A policy plan for the government is a sequence of functions \( \sigma = \{\sigma_t|t = 0, 1, 2, \ldots\} \), where \( \sigma_t(\xi_{t-1}) \) specifies money growth at time \( t \) conditional on the realization of history \( \xi_{t-1} \).

For any policy plan \( \sigma \) and any date \( t = 0, 1, 2, \ldots \), define the continuation of \( \sigma \) at \( t \) by \( \sigma^t = \{\sigma_{t+s}|s = 0, 1, 2, \ldots\} \).

For the representative firm, an allocation rule is now a sequence of functions \( \pi^i = \{\pi_i^j|t = 0, 1, 2, \ldots\} \), where \( \pi_i^j(\xi_{i-1}) \) specifies the choice of \( p_t(i) \) conditional on the realization of history \( \xi_{i-1} \). Likewise, for the representative household, an allocation rule is a sequence of functions \( h = \{h_t|t = 0, 1, 2, \ldots\} \), where \( h_t(\xi_t) \) specifies the choice of \( H_t \) conditional on \( \xi_t \). Note that these definitions recognize that firms set their prices for time \( t \) before the government chooses \( x_t \) and hence can condition only on \( \xi_{t-1} \). Households, on the other hand, make their time \( t \) decisions after observing \( x_t \) and can condition on \( \xi_t \). Given the allocation rules \( \pi^i \) and \( h \), define the continuation rules \( \pi^{lt} = \{\pi_{t+s}^i|s = 0, 1, 2, \ldots\} \) and \( h' = \{h_{t+s}|s = 0, 1, 2, \ldots\} \).

Without commitment, dividends \( d_t(i) = d_t(i, \xi_t) \), wage rates \( w_t = w_t(\xi_t) \), interest rates \( R_t = R_t(\xi_t) \), and the price level \( p_t = p_t(\xi_{t-1}) \) are also functions of the history of policy. Starting from any date \( t \) and history \( \xi_{t-1} \), however, private agents can forecast these variables perfectly using the recursive formula

\[
\xi_{t+s} = (\xi_{t+s-1}, \sigma_{t+s}(\xi_{t+s-1})), \quad s = 0, 1, 2, \ldots, \quad \text{and their knowledge of the functions}\]

\[
d_{t+s}(i) = [p_{t+s}(i) - w_{t+s}] \left[ \frac{p_{t+s}(i)}{p_{t+s}} \right]^{-\theta} (x_{t+s}/p_{t+s})
\]

for each \( s = 0, 1, 2, \ldots \), taking \( w_{t+s} = w_{t+s}(\xi_{t+s}) \), \( p_{t+s} = p_{t+s}(\xi_{t+s}) \), and \( \xi_{t+s} = (\xi_{t+s-1}, \sigma_{t+s}(\xi_{t+s-1})) \) as given for all \( s = 0, 1, 2, \ldots \).

\[\text{In general, these plans could depend on the history of firm and household decisions as well. But here, as in Chari and Kehoe (1990), only the history of policy matters: no firm or household perceives that its decisions influence those of the government or any other private agent.}\]
During each period \( t = 0, 1, 2, \ldots \), the representative household takes \( \xi_t, \sigma, \) and \( \pi^i \) for all \( i \in [0, 1] \) as given and chooses its continuation rule \( h^i \) to solve

\[
\text{(HT) Maximize} \\
\sum_{s=0}^{\infty} \beta^s \left( c_{t+s}^2 / \alpha - n_{t+s} \right) \\
\text{subject to} \\
c_{t+s} = \left[ \int_0^1 c_{t+s}(i)^{(\theta-1)/\theta} \, di \right]^{\theta/(\theta-1)} \\
\int_0^1 p_{t+s}(i)c_{t+s}(i)di + b_{t+s+1}x_{t+s}/R_{t+s} + m_{t+s+1}x_{t+s} \\
\leq m_{t+s} + (x_{t+s} - 1) + b_{t+s} + \int_0^1 d_{t+s}(i)di + w_{t+s}n_{t+s},
\]

and

\[
\int_0^1 p_{t+s}(i)c_{t+s}(i)di \leq m_{t+s} + (x_{t+s} - 1) + b_{t+s} - b_{t+s+1}x_{t+s}/R_{t+s}
\]

for each \( s = 0, 1, 2, \ldots \), taking \( d_{t+s}(i) = d_{t+s}(i, \xi_{t+s}), w_{t+s} = w_{t+s}(\xi_{t+s}), R_{t+s} = R_{t+s}(\xi_{t+s}), \) and \( \xi_{t+s+1} = (\xi_{t+s}, \sigma_{t+s+1}(\xi_{t+s})) \) as given for all \( i \in [0, 1] \) and \( s = 0, 1, 2, \ldots \).

In addition, for each \( t = 0, 1, 2, \ldots \) and \( \xi_t \), the continuation policy \( h^i \) must be consistent with the market clearing conditions

\[
\text{(MT) } m_{t+s+1} = 1, b_{t+s+1} = 0, \text{ and} \\
\int_0^1 c_{t+s}(i)di = n_{t+s}
\]

for all \( s = 0, 1, 2, \ldots \).

Finally, at each date \( t = 0, 1, 2, \ldots \), the government takes \( \xi_0, \pi^i \) for all \( i \in [0, 1] \), and \( h \) as given and chooses a continuation policy \( \sigma^i \) to solve

\[
\text{(GT) Maximize (13), where } c_{t+s} \text{ and } n_{t+s} \text{ are determined by } \pi^i \text{ for all } i \in [0, 1] \text{ and } h \text{ for all } s = 0, 1, 2, \ldots.
\]

As before, let \( \pi \) refer to the set of functions \( \pi^i \) for all \( i \in [0, 1] \). Then, following Chari and Kehoe (1990), the definition of a sustainable equilibrium describes outcomes that can prevail when the government lacks a commitment technology. Formally, a sustainable equilibrium consists of a policy plan \( \sigma \) and a set of
allocation rules \((\pi, h)\) that satisfy: (i) given \(\sigma, \pi^j\) for all \(j \in [0,1], j \neq i, \) and \(h,\) the continuation \(\pi^t\) of each \(\pi^t\) solves (FT) for all \(t = 0,1,2, \ldots\) and \(\xi_{t-1};\) (ii) given \(\sigma\) and \(\pi^t\) for all \(i \in [0,1],\) the continuation \(h^t\) of \(h\) solves (HT) for all \(t = 0,1,2, \ldots\) and \(\xi_t;\) (iii) the continuation \(h^t\) of \(h\) is consistent with (MT) for all \(t = 0,1,2, \ldots\) and \(\xi_t;\) (iv) given \(\pi^t\) for all \(i \in [0,1]\) and \(h,\) the continuation \(\sigma^t\) of \(\sigma\) solves (GT) for all \(t = 0,1,2, \ldots\) and \(\xi_{t-1}.\)

Associated with any sustainable equilibrium \((\sigma, \pi, h)\) is a sustainable outcome \((x, \Pi, H)\) defined as follows. Starting from \(\xi_{t-1} = \emptyset,\) construct \(\xi = \{\xi_t|t = 0,1,2, \ldots\}\) recursively using \(x_t = v_t(\xi_{t-1})\) and \(\xi_t = (\xi_{t-1}, \sigma_t(\xi_{t-1})).\) Then, for all \(t = 0,1,2, \ldots,\) construct \(\Pi = \{\Pi^t|i \in [0,1]\},\) \(\Pi^t = \{p_t(i)|t = 0,1,2, \ldots\},\) and \(H = \{H_t|t = 0,1,2, \ldots\}\) using \(p_t(i) = \pi_t^j(\xi_{t-1})\) for all \(i \in [0,1]\) and \(H_t = h_t(\xi_t).\) Thus, the sustainable outcome \((x, \Pi, H)\) describes the sequence of equilibrium prices and quantities that obtains when the government chooses \(x_t\) sequentially to maximize the representative household's utility and private agents respond optimally.

Chari and Kehoe (1990) characterize the set of sustainable outcomes by adapting Abreu's (1988) optimal penal codes to policy games played between a benevolent government and a large number of private agents. Following Chari and Kehoe, begin by defining the set of autarky plans \((\sigma^a, \pi^a, h^a).\) Let \(\sigma^a = \{\sigma^t_a|t = 0,1,2, \ldots\}\) have \(\sigma^a_t(\xi_{t-1}) = x\) for all \(t = 0,1,2, \ldots\) and \(\xi_{t-1}.\) Given \(\sigma^a,\) let each \(\pi^a, i \in [0,1],\) be the allocation rule such that its continuation \(\pi^{aut}_t\) solves (FT) for all \(t = 0,1,2, \ldots\) and \(\xi_{t-1}.\) Let \(h^a\) be the allocation rule such that its continuation \(h^{aut}_t\) solves (HT) and is consistent with (MT) for all \(t = 0,1,2, \ldots\) and \(\xi_t.\) Part 3 of the appendix establishes

**Proposition 4.1.** \((\sigma^a, \pi^a, h^a)\) is a sustainable equilibrium.

The sustainable outcome associated with \((\sigma^a, \pi^a, h^a),\) which has \(x_t = \bar{x}\) for all \(t = 0,1,2, \ldots,\) corresponds to Barro and Gordon's (1983) discretionary outcome, where the government takes private expectations as given. Here, as in Grossman (1990), there are costs of expected inflation; the government takes these as given as well. Thus, the government only accounts for the benefits of unanticipated inflation and sets \(x_t\) above its expected value in an effort to push equilibrium output closer to its efficient level. In equilibrium, however, private agents recognize that the government has this incentive, and adjust their expectations of money growth accordingly. Intuitively, if private expectations of money growth at the beginning of the current period are given by \(\bar{x},\) then the government has an incentive to increase output by setting actual money growth at \(x_t = x^g + \Delta_0,\) where \(0 < \Delta_0 < \infty.\) And if private agents respond by adjusting their expectations to \(x^g + \Delta_0,\) then the government has an incentive to set money growth still higher, at \(x_t = x^g + \Delta_0 + \Delta_1,\) where \(0 < \Delta_1 < \infty.\) This process continues until \(x_t\) reaches the upper bound \(\bar{x}.\) In fact, without the constraint \(x_t \leq \bar{x},\) money growth would be infinite; as in Calvo (1978) and Grossman (1990), the finite
upper bound on $x_t$ is needed to make the government's problem well-defined. In
the limit as $\bar{x}$ becomes arbitrarily large, (12) implies that the outcome associated
with the autarky plans does indeed become autarkic, with $c_t = n_t$ approaching
zero for all $t = 0, 1, 2, \ldots$.

By definition, any sustainable outcome is also an equilibrium under commit-
ment. Thus, it follows from Propositions 3.3 and 4.1 that $(\sigma^a, \pi^a, \sigma^h)$ generate
the worst sustainable outcome. As explained by Chari and Kehoe (1990), policy
plans and allocation rules that revert to this worst possible outcome form the
analog to Abreu's optimal penal codes and can be used to characterize outcomes
that yield higher levels of utility.

Given an arbitrary policy $x$ and an arbitrary allocation $(\Pi, H)$, define the cor-
responding revert-to-autarky plans $(\sigma'^r, \pi'^r, h'^r)$ as follows. For all $t = 0, 1, 2, \ldots$,
let $\sigma_t^r(\xi_{t-1}) = x_t$ for history $\xi_{t-1} = \{x_s | s = 0, 1, \ldots, t-1\}$; let $\sigma_t^r(\xi_{t-1}) = \bar{x}$ otherwise. For all $i \in [0, 1]$, let $\pi_t^r(\xi_{t-1}) = p_t(i)$ for $\xi_{t-1} = \{x_s | s = 0, 1, \ldots, t-1\}$; let $\pi_t^r(\xi_{t-1}) = \pi_t^a(\xi_{t-1})$ otherwise. Let $h_t^r(\xi_t) = H_t$ for $\xi_t = \{x_s | s = 0, 1, \ldots, t\}$. If $\xi_t = (\xi_{t-1}, \bar{x})$ is such that $\xi_{t-1} = \{x_s | s = 0, 1, \ldots, t-1\}$ but $\bar{x} \neq x_t$, let $h_t^r(\xi_t)$ be given by the continuation rule $h^r$ that solves (HT) given $\sigma^t$ and $\pi^t$ for all $i \in [0, 1]$. If $\xi_t = (\xi_{t-1}, \bar{x})$ is such that $\xi_{t-1} \neq \{x_s | s = 0, 1, \ldots, t-1\}$, let $h_t^r(\xi_t) = h_t^r(\xi_t)$. Thus, the revert-to-autarky plans dictate continuation with
$(x, \Pi, H)$ so long as the government has followed $x$ in the past; otherwise, they
revert to the autarky plans.

Next, let $U^a$ denote the constant value of the household’s utility $c_t^a/\alpha - n_t$ when
c_t and n_t are generated by the autarky plans $(\sigma^a, \pi^a, \sigma^h)$; using (12) with $\bar{x} = \bar{x}$

$$U^a = (1/\alpha)[\beta(\theta - 1)/\theta \bar{x}]^{\alpha/(1-\alpha)} - [\beta(\theta - 1)/\theta \bar{x}]^{1/(1-\alpha)}.$$  

For any $p_t$, define $U^d(p_t)$ by

$$U^d(p_t) = \max_{\bar{x} \in [\beta, \bar{x}]} (\bar{x}/p_t)^{\alpha} - (\bar{x}/p_t).$$ (GD)

When each firm $i \in [0, 1]$ sets its prices according to $\Pi^f = \{p_t(i) | t = 0, 1, 2, \ldots\}$,
p_t is given by

$$p_t = \left[ \int_0^1 p_t(i)^{1-\theta} \, dx \right]^{1/(1-\theta)}.$$ (14)

Thus, as shown in part 4 of the appendix, $U^d(p_t)$ is the maximum current-period
utility that the government can obtain by deviating from $x$ at time $t$, given that it
has followed $x$ in every period prior to $t$, when private agents follow the revert-
to-autarky plans associated with $(x, \Pi, H)$. Part 4 of the appendix also establishes

**Proposition 4.2.** Let $x$ be an arbitrary policy and $(\Pi, H)$ an arbitrary allocation. Then $(x, \Pi, H)$ is the outcome of a sustainable equilibrium if and only if:

(i) $(x, \Pi, H)$ is an equilibrium under commitment;
(ii) \((x, \Pi, H)\) satisfies

\[
U^d(p_t) + \left[ \beta/(1 - \beta) \right] U^a \leq \sum_{s=0}^{\infty} \beta^s (c^a_{t+s}/\alpha - n_{t+s})
\]

for all \(t = 0, 1, 2, \ldots\), where \(c_t\) and \(n_t\) are given by \(H\) and where \(p_t\) is determined by \(\Pi\) and (14) for all \(t = 0, 1, 2, \ldots\).

Eq. (15) indicates that, at each date \(t = 0, 1, 2, \ldots\), any sustainable equilibrium outcome must provide a utility level from date \(t\) forward that is at least as large as the utility level obtained by deviating at date \(t\) and reverting to autarky thereafter. Thus, it resembles conditions, derived by Barro and Gordon (1983), Rogoff (1989), and Grossman (1990), that allow outcomes to be supported by simple trigger-strategy equilibria. However, Proposition 4.2 provides a complete characterization of all sustainable outcomes, including those that are supported by more complicated reputational equilibria.

5. Sustainability of the Friedman rule

Proposition 4.2 lists the full set of conditions that must be satisfied for the optimal policy under commitment, the Friedman rule, to be part of a sustainable outcome. If \((x, \Pi, H)\) is an equilibrium under commitment with \(x_t = \beta\) for all \(t = 0, 1, 2, \ldots\), then (7), (8), and (10) imply that this equilibrium must have \(p_t = p\) and \(c_t = n_t = \bar{c}\) for all \(t = 0, 1, 2, \ldots\), where

\[
p = \beta[\theta/(\theta - 1)]^{1/(1-\alpha)}
\]

and

\[
\bar{c} = [(\theta - 1)/\theta]^{1/(1-\alpha)}.
\]

With \(p_t = p\), the solution to (GD) has

\[
U^d(p) = (c^*)^\alpha/\alpha - c^*,
\]

where \(c^* = 1\) is the efficient level of output.\(^3\) Finally, the outcome under the autarky plans \((\sigma^a, \pi^a, h^a)\) has \(x_t = \bar{x}\) and, by (7), (8), and (10), \(c_t = n_t = c\) for all \(t = 0, 1, 2, \ldots\), where

\[
c = [\beta(\theta - 1)/\theta \bar{x}]^{1/(1-\alpha)},
\]

so that

\[
U^a = c^2/\alpha - c.
\]

\(^3\) This solution requires that \(\bar{x}\) be sufficiently large. Specifically, \(\beta[\theta/(\theta - 1)]^{1/(1-\alpha)} \leq \bar{x}\) must hold.
Thus, when \( x \) is given by the Friedman rule, (15) specializes to
\[
(1 - \beta)[(c^*)^\alpha / \alpha - c^*] + \beta(c^*)^\alpha / \alpha - c) \leq \bar{c}^\alpha / \alpha - \bar{c}.
\]  

(16)

By following the Friedman rule, the government removes the distortion caused by a binding cash-in-advance constraint, keeping output at level \( \bar{c} \). By deviating from the Friedman rule, the government creates a positive monetary surprise that increases output to its efficient level \( c^* > \bar{c} \), thereby eliminating the distortion caused by monopolistic competition as well. This deviation destroys the government’s reputation, however. Private agents revert to the autarky plans \((\pi^a, h^a)\), inducing the government to inflate at the highest possible rate thereafter; output falls to \( c < \bar{c} \). Hence, the Friedman rule can be supported without commitment if and only if (16) holds.

Since neither \( c \) nor \( c^* \) depends on \( \bar{x} \), and since \( c \) is strictly decreasing in \( \bar{x} \), (16) implies that the Friedman rule is more likely to be sustainable when \( \bar{x} \) is large. As \( \bar{x} \) increases, the penalty imposed by reversion to autarky gets more severe; the government becomes more willing to adhere to the Friedman rule to avoid this penalty.

Since the constraint \( x_t \leq \bar{x} \) is imposed for purely technical reasons, consider what happens in the limiting case where \( \bar{x} \) becomes arbitrarily large. As \( \bar{x} \) approaches zero, (16) reduces to
\[
(1 - \beta)[(1 / \alpha - 1)](\theta - 1)/\theta^{\alpha/(1 - \alpha)} - [(\theta - 1)/\theta]^{1/(1 - \alpha)}.
\]

(17)

This condition becomes more likely to hold as \( \beta \) increases. Moreover, the right-hand side is strictly positive, while the left-hand side converges to zero as \( \beta \) approaches unity. Hence, the Friedman rule can always be sustained for sufficiently large values of \( \beta \). A rise in the discount factor increases the cost of permanently reverting to autarky; the government has less incentive to deviate when \( \beta \) is close to one. Barro and Gordon (1983), Rogoff (1989), and Grossman (1990) obtain similar results.

Eq. (2) indicates that when \( \theta \) rises, goods become closer substitutes. Thus, markets become more competitive; the distortion caused by monopolistic competition shrinks. Since this distortion gives the government an incentive to create monetary surprises, (17) implies that the government becomes less likely to deviate from the Friedman rule when \( \theta \) is large. In the limit as \( \theta \) approaches infinity, markets become perfectly competitive, and (17) holds for all \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \). This result is also noted by Blackburn and Christensen (1989) and Rogoff (1989): the time consistency problem for monetary policy vanishes when the nonmonetary distortion is removed.

The derivative of (17) with respect to \( \alpha \) is ambiguous in sign. Numerical analysis reveals, however, that (17) becomes more likely to hold as \( \alpha \) decreases. Smaller values of \( \alpha \) make utility more concave, thereby increasing the costs of
reverting to autarky relative to the current-period gain from deviating. Thus, the government's incentive to adhere to the Friedman rule gets stronger as \( x \) falls.\(^4\)

Barro and Gordon (1983), Rogoff (1989), and Grossman (1990) derive conditions similar to (17). Since their models start with ad hoc specifications for the government's objective function, however, they provide little guidance for assigning values to the key parameters. Here, where government and private objectives coincide, the parameters of (17) come directly from the representative household's utility function; values for these key parameters can be found by consulting previous work with calibrated models.

Thus, consider an annual model, with \( \beta = 0.95 \). Since (10) indicates that \( \theta/(\theta - 1) \) is the ratio of price to marginal cost, a value for \( \theta \) can be drawn from the literature on markups. Based on their review of this literature, Rotemberg and Woodford (1992) select a value of 1.2 for the markup, although they consider alternative values between 1 and 2. As indicated above, the Friedman rule becomes more difficult to support in a sustainable equilibrium as the markup rises. Thus, the value at the top of Rotemberg and Woodford's range provides the Friedman rule with its greatest challenge.

With \( \beta = 0.95 \) and \( \theta/(\theta - 1) = 2 \), (17) holds for all \( 0 < \alpha < 0.864 \). With \( \beta = 0.95 \) and \( \theta/(\theta - 1) = 1.2 \), (17) holds for all \( 0 < \alpha < 0.962 \). Thus, the Friedman rule can be supported without commitment for almost all values of \( \alpha \). Only in extreme cases, where preferences are nearly linear, does the Friedman rule fail to be sustainable.

Even when the Friedman rule is sustainable, however, Proposition 4.2 implies that there exist other sustainable equilibria with higher money growth rates. As a final example that illustrates the existence of multiple equilibria, let \( \alpha = 0.5 \), \( \beta = 0.95 \), \( \theta/(\theta - 1) = 2 \), and \( \bar{x} = 100 \). In this case, numerical analysis reveals that all constant money growth rates between \( \beta \) and \( \bar{x} \) can be supported in sustainable equilibria.

6. Conclusion

As surveys by Blackburn and Christensen (1989) and Rogoff (1989) make clear, Barro and Gordon's (1983) model has proven useful in addressing a wide range of questions related to the time consistency of monetary policy. The Barro–Gordon framework, however, is not developed from an environment that is fully specified at the level of preferences, endowments, and technologies; instead, it starts with a pair of equations that describe the government's

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\(^4\) With \( \alpha \leq 0 \), in fact, utility under autarky approaches negative infinity, so that the Friedman rule is always sustainable, as \( \bar{x} \) gets arbitrarily large. For this reason, the case \( 0 < \alpha < 1 \) analyzed here is more interesting.
objectives along with the behavior of the aggregate economy. This paper fills a gap in the literature by demonstrating that a version of the Barro–Gordon problem emerges from a dynamic general equilibrium model with utility maximizing households, monopolistically competitive firms, and sticky goods prices.

By explicitly linking the government's objectives to those of the private sector, the model developed here implies that the key parameters are those of a representative household's utility function, rather than those of an ad hoc government loss function. Values for these key parameters, therefore, can be drawn from previous work with calibrated models. The results demonstrate that for reasonable sets of parameter values, the optimal policy under commitment, given by the Friedman (1969) rule, can also be supported in a reputational equilibrium without commitment.

As in most of the time consistency literature, the existence of a multiple equilibria is an unresolved problem here. Even when the results indicate that the Friedman rule can be supported in an equilibrium without commitment, they also imply the existence of many other equilibria featuring higher levels of inflation and lower levels of welfare. Determining which of these equilibria is most likely to prevail and how the government might successfully choose between these equilibria are important tasks for future research.

Appendix A

A.1. Proof of Proposition 3.1

In a Ramsey equilibrium, the government chooses a policy \( x = \{x_t | t = 0, 1, 2, \ldots \} \) with \( x_t \in [\beta, \bar{x}] \) for all \( t = 0, 1, 2, \ldots \) to maximize (1) subject to the requirements that equilibrium prices and quantities are determined by allocation rules \((\pi, h)\) that solve (FO) and (HO) and are consistent with (MO). Proposition 3.1 is established by showing that these requirements are summarized by (9), (11), and \( c_t = n_t \) for all \( t = 0, 1, 2, \ldots \).

The problem (HO) dictates that the representative household choose sequences for \( c_t, c_t(i), n_t, m_{t+1}, \) and \( b_{t+1} \) to maximize (1) subject to (2), (4), and (5), taking as given the sequences for \( d_t(i), w_t, R_t, x_t, \) and \( p_t(i) \). Under the additional assumption that the cash-in-advance constraint (5) holds with equality, even when it does not bind, necessary and sufficient conditions for a solution to this problem are (2), (4), and

\[
c_t^{\alpha-1+1/\theta} c_t(i)^{-1/\theta} = (\lambda_t + \mu_t) p_t(i), \quad (A.1)
\]

\[
1 = \lambda_t w_t, \quad (A.2)
\]
\[ \lambda_t x_t = \beta(\lambda_{t+1} + \mu_{t+1}), \quad (A.3) \]

\[ (\lambda_t + \mu_t)x_t/R_t = \beta(\lambda_{t+1} + \mu_{t+1}), \quad (A.4) \]

\[ x_t = \int_0^1 p_t(i)c_t(i)di, \quad (A.5) \]

and

\[ \lim_{t \to \infty} \beta^t(\lambda_t + \mu_t)m_t = 0, \quad (A.6) \]

where \( \lambda_t \) and \( \mu_t \) are nonnegative multipliers on the budget constraint (4) and the cash-in-advance constraint (5) and where the market clearing conditions \( m_t = 1 \) and \( b_t = b_{t+1} = 0 \) from (MO) have been imposed in deriving (A.5).

Using (14), Eqs. (2), (A.1), and (A.5) imply

\[ \lambda_t + \mu_t = c_t^\alpha/x_t, \quad (A.7) \]

Substituting (A.7) back into (A.1) and using (2) and (14) yields

\[ c_t = x_t/p_t \quad (A.8) \]

and

\[ c_t(i) = [p_t(i)/p_t]^\theta(x_t/p_t). \quad (A.9) \]

Note that (A.9), describing the representative household's demand for good \( i \), can be rewritten as (3) in the text.

Combining (A.4), (A.7), and (A.8) yields

\[ R_t = (x_{t+1}/\beta)(x_t/p_t)\theta(x_{t+1}/p_{t+1})^{-\alpha}. \quad (A.10) \]

Finally, (A.2), (A.3), (A.7), and (A.8) imply

\[ w_t = (1/\beta)x_t x_{t+1}(p_{t+1}/x_{t+1})^\alpha, \quad (A.11) \]

which corresponds to (8) in the text.

Next, consider the representative firm's problem (FO): maximize (6), taking \( w_t, p_t, \) and \( x_t \) as given. The unique solution to this problem has

\[ p_t(i) = [\theta/(\theta - 1)]w_t, \quad (A.12) \]

Eq. (A.12) shows that in equilibrium, all firms set the same price. Hence, using (14),

\[ p_t(i) = p_t = [\theta/(\theta - 1)]w_t \quad (A.13) \]

for all \( i \in [0, 1] \), which is (10) in the text. Eqs. (A.8) and (A.9), along with the condition

\[ \int_0^1 c_t(i)di = n_t \]
from (MO), then imply
\[ c_i(i) = c_t = n_t = x_t/p_t \quad (A.14) \]
for all \( i \in [0,1] \), which yields (7) in the text.

Combining (A.6), (A.7), and \( m_t = 1 \) yields (9). Combining (A.11), (A.13), and (A.14) yields (11) and \( c_t = n_t \) for all \( t = 0,1,2,\ldots \). This shows that (MO) and the solutions to (FO) and (HO) imply (9), (11), and \( c_t = n_t \) for all \( t = 0,1,2,\ldots \).

Now let \( c_t = n_t \) satisfy (9) and (11) for all \( t = 0,1,2,\ldots \). Given this sequence for \( c_t \), construct sequences for \( c_t(i), m_{t+1}, b_{t+1}, p_t(i), p_t, w_t, d_t(i), R_t, \lambda_t, \) and \( \mu_t \) using
\[ c_t(i) = c_t, \quad m_{t+1} = 1, \quad b_{t+1} = 0, \quad p_t(i) = p_t - x_t/c_t, \quad w_t = [\theta/(\theta - 1)]p_t, \quad d_t(i) = (p_t - w_t)c_t, \quad R_t = (x_{t+1}/\beta)(c_t/c_{t+1})^\alpha, \quad \lambda_t + \mu_t = c_t^\alpha/x_t, \]
and
\[ \lambda_t = [\theta/(\theta - 1)](1/p_t). \]

Equations (2), (4), and (A.1)–(A.6) imply that these sequences for \( c_t, c_t(i), n_t, m_{t+1}, \) and \( b_{t+1} \) solve (HO). Eq. (A.12) implies that this sequence for \( p_t(i) \) solves (FO). Finally, these sequences are consistent with (MO). This shows that if \( c_t = n_t \) satisfies (9) and (11) for all \( t = 0,1,2,\ldots \), then it is consistent with (MO) and solutions to (FO) and (HO).

Thus, (9), (11), and \( c_t = n_t \) for all \( t = 0,1,2,\ldots \) completely summarize the restrictions imposed on \( c_t \) and \( n_t \) by allocation rules \( (\pi, h) \) that solve (FO) and (HO) and are consistent with (MO).

A.2. Proof of Propositions 3.2 and 3.3

According to Proposition 3.1, the Ramsey outcome can be constructed by maximizing (1) subject to (9), (11), \( c_t = n_t, \) and \( x_t \in [\beta, \bar{x}] \) for all \( t = 0,1,2,\ldots \). Eq. (11) is equivalent to
\[ \ln(c_t) = \ln[\beta(\theta - 1)/\theta] + \alpha \ln(c_{t+1}) - \ln(x_{t+1}). \]
Since \( 0 < \alpha < 1 \) and \( x_{t+1} \) is bounded, this linear difference equation in \( \ln(c_t) \) can be solved forward to obtain
\[ \ln(c_t) = \{\ln[\beta(\theta - 1)/\theta]\}/(1 - \alpha) - \sum_{j=0}^{\infty} \alpha^j \ln(x_{t+j+1}). \]
From this expression, it is apparent that the constraints $x_t \in [\beta, \bar{x}]$ on money growth translate into the constraints $c_t \in [\underline{c}, \bar{c}]$ on consumption, where

$$c_t = \frac{\beta(\theta - 1)/\theta \bar{x}}{1/(1 - \alpha)}$$

and

$$\bar{c} = \left(\frac{1 - \alpha}{1/\theta}\right)^{1/(1 - \alpha)}.$$

Thus, the problem of finding the Ramsey outcome reduces to one of finding a sequence for $c_t$ to maximize

$$\sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to $c_t \in [\underline{c}, \bar{c}]$ for all $t = 0, 1, 2, \ldots$, where

$$U(c_t) = \frac{c_t^{\alpha}/\alpha - c_t}{c_t^{\alpha}/\alpha}.$$

The following lemma implies that the solution to this problem has $c_t = \bar{c}$ for all $t = 0, 1, 2, \ldots$.

**Lemma.** $U(c_t)$ is strictly increasing on $[\underline{c}, \bar{c}]$.

**Proof.** Note first that $U''(c_t) < 0$ for all $c > 0$. In addition, $U'(c^*) = 0$, where $c^* = 1$. Since $c^* > \bar{c}$, it follows that $U'(c_t) > 0$ for all $c_t \in [\underline{c}, \bar{c}]$.

With $c_t = \bar{c}$ for all $t = 0, 1, 2, \ldots$, (11) implies that $x_{t+1} = \beta$ for all $t = 0, 1, 2, \ldots$. Equilibrium quantities do not depend on $x_0$; under commitment, the choice of $x_0$ amounts to a redefinition of nominal units. Hence, it may be assumed that $x_0 = \beta$ as well. Thus, the Ramsey outcome has $x_t = \beta$ for all $t = 0, 1, 2, \ldots$. Substituting $x_{t+1} = \beta$ and $c_t = \bar{c}$ into (A.8) and (A.10) yields $R_t = 1$ for all $t = 0, 1, 2, \ldots$, completing the proof of Proposition 3.2.

The proof of Proposition 3.3 mirrors that of Proposition 3.2. The lemma implies that utility is minimized with $c_t = \underline{c}$ for all $t = 0, 1, 2, \ldots$, which requires that $x_{t+1} = \bar{x}$ for all $t = 0, 1, 2, \ldots$. Since $x_0$ can be chosen arbitrarily, the policy $x_t = \bar{x}$ for all $t = 0, 1, 2, \ldots$ yields the worst equilibrium under commitment.

**A.3. Proof of Proposition 4.1**

Under $\sigma^d$, the representative firm takes $\xi_{t-1}$ as given at each date $t = 0, 1, 2, \ldots$ and solves (FT) assuming $x_{t+s} = \bar{x}$ for all $s = 0, 1, 2, \ldots$. Similarly, the representative household takes $\xi_t$ as given at each date $t = 0, 1, 2, \ldots$ and solves (HT) assuming $x_{t+s+1} = \bar{x}$ for all $s = 0, 1, 2, \ldots$.

For $t = 0, 1, 2, \ldots$, the necessary and sufficient conditions for solutions to (FT) and (HT) coincide with those for (FO) and (HO). Thus, substitute (A.11) into
(A.13) with \( x_t = x_{t+1} = \bar{x} \) to obtain a difference equation with solution \( p_t = \bar{\rho} \) for all \( t = 0, 1, 2, \ldots \), where

\[
\bar{\rho} = \left\{ \left(1/\beta\right)[\theta/((\theta - 1))]\bar{x}^{2-a}\right\}^{1/(1-a)}.
\]  

(A.15)

It follows that each \( \pi^a_i, i \in [0, 1] \), has \( \pi^a_i(\xi_{t-1}) = \bar{\rho} \) for all \( t = 0, 1, 2, \ldots \) and \( \xi_{t-1} \). Substituting this solution into (A.14) reveals that \( h^a_t(\xi_t) \) must have

\[
c_t = n_t = x_t/\bar{\rho}
\]

for all \( t = 0, 1, 2, \ldots \) and \( \xi_t \).

By construction, \( \pi^a_i \) for all \( i \in [0, 1] \) and \( h^a \) satisfy conditions (i)–(iii) in the definition of a sustainable equilibrium. It only remains to verify condition (iv): given \( \pi^a_i \) for all \( i \in [0, 1] \) and \( h^a \), the continuation \( \sigma^u_t \) of \( \sigma^a \) must solve (GT) for all \( t = 0, 1, 2, \ldots \) and \( \xi_{t-1} \). In light of (A.16), this condition is satisfied so long as

\[
\bar{x} = \arg \max_{x \in [\beta, \bar{x}]} \left( \bar{x}/\bar{\rho} \right)^{\alpha} - (\bar{x}/\bar{\rho}).
\]  

(A.17)

With \( \bar{\rho} \) given by (A.15), the objective function in (A.17) is strictly increasing on \([\beta, \bar{x}]\). Thus, (A.17) holds, as does condition (iv), so that \( (\sigma^a, \pi^a, h^a) \) is a sustainable equilibrium.

A.3. Proof of Proposition 4.2

First, suppose that \((x, \Pi, H)\) is a sustainable outcome, associated with some sustainable equilibrium \((\sigma, \pi, h)\). Since (FT), (HT), and (MT) at \( t = 0 \) coincide with (FO), (MO), and (HO), conditions (i)–(iii) in the definition of a sustainable equilibrium imply that \((x, \Pi, H)\) is an equilibrium under commitment.

Since \( h \) is part of a sustainable equilibrium, its continuation \( h^t_t \) must solve (HT) and satisfy (MT) at each date \( t = 0, 1, 2, \ldots \). Since, for \( t = 0, 1, 2, \ldots \), the necessary and sufficient conditions for (HT) coincide with those for (HO) and the market clearing conditions in (MT) coincide with those in (MO), (A.14) implies that \( h_t(\xi_t) \) must have

\[
c_t = n_t = x_t/p_t
\]

for all \( t = 0, 1, 2, \ldots \) and \( \xi_t \).

Thus, given \( t \) and \( \xi_{t-1} \), the government could deviate from \( x \) at time \( t \), thereby obtain the current-period utility \( U^d(p_t) \) defined by (GD) in the text, and follow the autarky plan \( \sigma^a \) thereafter. This deviation yields the total utility given by the left-hand side of (15). Since condition (iv) in the definition of a sustainable equilibrium requires that the government not have any incentive to deviate, (15) must hold.

Next, let \((x, \Pi, H)\) be an equilibrium under commitment that satisfies (15), and let \((\sigma^f, \pi^f, h^f)\) be the associated revert-to-autarky plans. The proof is completed by showing that \((\sigma^f, \pi^f, h^f)\) is a sustainable equilibrium.
Consider histories $\xi_{t-1}$ along which the government has not deviated from $x$. Since $(x, \Pi, H)$ is an equilibrium under commitment, $H$ solves (HO) and satisfies (MO) given $x$ and $\Pi^j$ for all $i \in [0,1]$, while each $\Pi^j$ solves (FO) given $x$, $\Pi^j$ for all $j \in [0,1]$, $j \neq i$, and $H$. Hence, the continuation of $H$ solves (HT) and satisfies (MT) at time $t-1$ given $x$ and $\Pi^j$ for all $i \in [0,1]$, and the continuation of $\Pi^j$ solves (FT) at time $t$ given $x$, $\Pi^j$ for all $j \in [0,1]$, $j \neq i$, and $H$. Neither the household nor the firm has an incentive to deviate after $\xi_{t-1}$.

Now consider the government’s incentive to deviate from $x$ at time $t$ following history $\xi_{t-1}$. Since $H$ solves (HO) and satisfies (MO), (A.14) implies that for all $\xi_t = (\xi_{t-1}, \bar{x})$, $h^j_t(\xi_t)$ has

$$c_t = n_t = \bar{x}/p_t.$$

Hence, using the proof of Proposition 4.1, the best the government can do by deviating is to choose $\bar{x}$ to solve (GD) at time $t$ and, faced with the autarky plans $(\pi^a, h^a)$, choose to follow $\sigma^a$ thereafter. This best deviation yields total utility given by the left-hand side of (15); since (15) holds, the government has no incentive to deviate from $x$ after $\xi_{t-1}$.

For histories $\xi_{t-1}$ along which the government has deviated from $x$, the revert-to-autarky plans specify that the government, the representative firm, and the representative household all follow the autarky plans $(\sigma^a, \pi^a, h^a)$ forever. By the proof of proposition 4.1, no agent has an incentive to deviate from these plans.

Thus, $(\sigma^a, \pi^a, h^a)$ is a sustainable equilibrium, and $(x, \Pi, H)$ is the associated sustainable outcome.

References