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A method for taking models to the data

Peter N. Ireland^{a,b,*}

^a*Department of Economics, Boston College, 140 Commonwealth Avenue, Chestnut Hill, MA 02467-3806, USA*

^b*National Bureau of Economic Research, Cambridge, MA 02138, USA*

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Abstract

This paper develops a method for combining the power of a dynamic, stochastic, general equilibrium model with the flexibility of a vector autoregressive time-series model to obtain a hybrid that can be taken directly to the data. It estimates this hybrid model via maximum likelihood and uses the results to address a number of issues concerning the ability of a prototypical real business cycle model to explain movements in aggregate output and employment in the postwar US economy, the stability of the real business cycle model's structural parameters, and the performance of the hybrid model's out-of-sample forecasts.

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1. Introduction

Two distinct approaches to macroeconomic analysis emerged during the early 1980s and continue to yield insights today. First, work following Sims (1980) characterizes and attempts to explain the movements and co-movements of key aggregate variables using vector autoregressive (VAR) time-series models. Second, work following Kydland and Prescott (1982) characterizes and attempts to explain the movements and co-movements of many of the same variables using dynamic, stochastic, general equilibrium (DSGE) models.

These two distinct approaches to macroeconomics both have their distinct strengths and weaknesses. VAR models, for instance, can be taken directly to the data: they

* Corresponding author. Department of Economics, Boston College, 140 Commonwealth Avenue, Chestnut Hill, MA 02467-3806, USA. Tel.: +1-617-552-3687; fax: +1-617-552-2308.

E-mail address: irelandp@bc.edu (P.N. Ireland).

are easy to estimate and, once estimated, can be used to perform statistical hypothesis tests as well as to generate out-of-sample forecasts. Moreover, since their specification requires little, if any, reference to detailed economic theory, VAR models remain flexible enough to address a wide range of questions regarding the nature and sources of business cycle fluctuations. Because they rely so loosely on economic theory, however, VAR models often fail to uncover parameters that are truly structural; thus, these models may exhibit instability across periods when monetary and fiscal policies change. Indeed, [Stock and Watson \(1996\)](#) find evidence of widespread instability in VAR models estimated with postwar US data.

DSGE models, by contrast, are firmly grounded in economic theory. These models draw tight links between the structural parameters describing private agents' tastes and technologies and the time-series behavior of endogenous variables such as aggregate output and employment; in principle, at least, these structural parameters should remain invariant to changes in policy regimes. Yet because they rely so heavily on economic theory, DSGE models are often regarded as being too stylized to be taken directly to the data, making traditional econometric methods for estimation, hypothesis testing, and forecasting inapplicable. Moreover, since they take such a strong stance on so many details concerning the structure of the economy, DSGE models often yield results that appear to be fragile, at least at first glance. When [Kydland and Prescott \(1982\)](#) report, for instance, that technology shocks can account for most of the observed output variation in the postwar US data, one is still left to wonder whether this result will survive modifications to their model, such as the introduction of other types of shocks.

This paper develops a method for combining the power of DSGE theory with the flexibility of VAR time-series models, in hopes of obtaining a hybrid that shares the desirable features of both approaches to macroeconomics. The method takes as its starting point a fully-specified DSGE model, but also admits that while this model may be powerful enough to account for and explain many key features of the US data, it remains too stylized to possibly capture all of the dynamics that can be found in the data. Hence, it augments the DSGE model so that its residuals – meaning the movements in the data that the theory cannot explain – are described by a VAR, making estimation, hypothesis testing, and forecasting feasible.

To illustrate how this method works, the rest of the paper unfolds as follows. The next section outlines a prototypical DSGE model: [Hansen's \(1985\)](#) real business cycle model with indivisible labor. Section 3 then augments this model with VAR residuals to arrive at the hybrid specification described above. Section 4 estimates the hybrid model via maximum likelihood and uses the estimated model to address a number of key issues concerning the ability of the real business cycle model to explain movements in output, consumption, investment, and hours worked in the postwar US data, the stability of the real business cycle model's structural parameters, and the performance of the hybrid model's out-of-sample forecasts. Section 5 summarizes and concludes, while two appendices provide additional information for those readers who are especially interested in the technical details behind the construction and estimation of the hybrid model.

Before moving on, however, mention should be made of some previous research that relates to the work presented here. [Christiano \(1988\)](#), [Altug \(1989\)](#), [Bencivenga](#)

(1992), McGrattan (1994), Hall (1996), Ireland (1997, 2001a, b, 2002), McGrattan et al. (1997), Chow and Kwan (1998), DeJong et al. (2000a, b), Kim (2000), and Schorfheide (2000) also estimate the structural parameters of DSGE models using maximum likelihood methods. In fact, a number of these studies – Altug (1989), McGrattan (1994), Hall (1996), and McGrattan et al. (1997) – draw on a framework for adding error terms to the structural equations of DSGE models developed originally by Sargent (1989). Sargent's (1989) ideas also lie at the heart of the approach taken here; thus, the connections between these earlier studies and the work presented here are discussed in more detail below.

Work with structural VAR models, following Bernanke (1986), Blanchard and Watson (1986), and Sims (1986), also attempts to draw on the power of economic theory while retaining the flexibility of more conventional VAR models; indeed, McKibbin et al. (1998) also refer to their structural VAR as a hybrid model. The goal of research with structural VAR models is the same as the goal of the work presented here: to develop macroeconomic models that have theoretical foundations and that are also successful in fitting the data. Structural VAR models, however, typically rely on economic theory only to the extent that it is absolutely required for identification, while the hybrid model constructed here is built around a fully-specified DSGE model. Which approach one prefers, therefore, depends on how much confidence one has in the underlying theory.

Finally, Smith (1993), Canova et al. (1994), Kim (2000), Schorfheide (2000), and Canova (2002) evaluate DSGE models by comparing their quantitative implications to those of unconstrained VAR models. Each of these earlier studies stops short, however, of suggesting how features of these two classes of macroeconomic models might be combined to obtain hybrids like the one considered here.

2. A prototypical DSGE model

In Hansen's (1985) real business cycle model with indivisible labor, a representative consumer has preferences defined over consumption C_t and hours worked H_t during each period $t = 0, 1, 2, \dots$, as described by the expected utility function

$$E \sum_{t=0}^{\infty} \beta^t [\ln(C_t) - \gamma H_t], \quad (1)$$

where the discount factor satisfies $1 > \beta > 0$ and where $\gamma > 0$. The linearity of utility in hours worked can be motivated, following Hansen (1985) and Rogerson (1988), by assuming that the economy consists of many individual consumers, each of whom either works full time or remains unemployed. The representative consumer produces output Y_t with capital K_t and labor H_t according to the constant-returns-to-scale technology described by

$$Y_t = A_t K_t^\theta (\eta^t H_t)^{1-\theta}, \quad (2)$$

where $\eta > 1$ measures the gross rate of labor-augmenting technological progress and where $1 > \theta > 0$. The technology shock A_t follows the first-order autoregressive

process:

$$\ln(A_t) = (1 - \rho) \ln(A) + \rho \ln(A_{t-1}) + \varepsilon_t, \quad (3)$$

where $A > 0$ and $1 > \rho > -1$. The serially uncorrelated innovation ε_t is normally distributed with mean zero and standard deviation σ .

During each period $t = 0, 1, 2, \dots$, the representative consumer divides output Y_t between consumption C_t and investment I_t , subject to the resource constraint

$$Y_t = C_t + I_t. \quad (4)$$

By investing I_t units of output during period t , the consumer increases the capital stock K_{t+1} available during period $t + 1$ according to

$$K_{t+1} = (1 - \delta)K_t + I_t, \quad (5)$$

where the depreciation rate satisfies $1 > \delta > 0$.

Equilibrium allocations for this economy solve the representative consumer's problem: choose sequences $\{Y_t, C_t, I_t, H_t, K_{t+1}\}_{t=0}^{\infty}$ to maximize the utility function (1) subject to constraints (2)–(5) for all $t = 0, 1, 2, \dots$. The first-order conditions for this problem include

$$\gamma C_t H_t = (1 - \theta) Y_t, \quad (6)$$

and

$$1/C_t = \beta E_t \{ (1/C_{t+1}) [\theta (Y_{t+1}/K_{t+1}) + 1 - \delta] \} \quad (7)$$

for all $t = 0, 1, 2, \dots$. The intratemporal efficiency condition (6) equates marginal rate of substitution between consumption and leisure to the marginal product of labor, while the intertemporal efficiency condition (7) equates the marginal rate of intertemporal substitution to the marginal product of capital.

Eqs. (2)–(7) form a system of six non-linear stochastic difference equations in the model's six variables: Y_t , C_t , I_t , H_t , K_t , and A_t . Approximate solutions to this system can be constructed as follows. Define $y_t = Y_t/\eta^t$, $c_t = C_t/\eta^t$, $i_t = I_t/\eta^t$, $h_t = H_t$, $k_t = K_t/\eta^t$, and $a_t = A_t$. Eqs. (2)–(7) imply that in the absence of technology shocks, when $\varepsilon_t = 0$ for all $t = 0, 1, 2, \dots$, the economy converges to a steady state in which each of these detrended variables remains constant, with $y_t = y$, $c_t = c$, $i_t = i$, $h_t = h$, $k_t = k$, and $a_t = a$. Appendix A shows exactly how the steady-state values y , c , i , h , k , and a depend on some of the model's underlying parameters describing tastes and technologies: β , γ , θ , η , δ , and A .

The Appendix A also describes how (2)–(7) can be log-linearized about this steady state. The methods of Blanchard and Kahn (1980), when applied to the linear system, then provide an approximate solution of the form

$$\mathbf{s}_t = \mathbf{A}\mathbf{s}_{t-1} + \mathbf{B}\varepsilon_t \quad (8)$$

and

$$\mathbf{f}_t = \mathbf{C}\mathbf{s}_t \quad (9)$$

for all $t = 0, 1, 2, \dots$, where the vectors \mathbf{s}_t and \mathbf{f}_t keep track of logarithmic, or percentage, deviations of each detrended variable from its steady-state level, with

$$\mathbf{s}_t = [\ln(k_t/k) \ln(a_t/a)]'$$

and

$$\mathbf{f}_t = [\ln(y_t/y) \ln(c_t/c) \ln(i_t/i) \ln(h_t/h)]'.$$

In (8) and (9), the elements of the matrices \mathbf{A} , \mathbf{B} , and \mathbf{C} also depend on some of the model's structural parameters: β , θ , η , δ , and ρ .

3. The hybrid model

In principle, one could use data on aggregate output, consumption, investment, and hours worked, along with the solution described by (8) and (9), to estimate the real business cycle model's structural parameters. Many researchers, however, including [Kydland and Prescott \(1982\)](#), argue that models of this type are too stylized to explain many features of the data, making traditional econometric methods inapplicable. Indeed, one dimension along which the real business cycle model is quite stylized lies in its assumption that just one shock – the aggregate technology shock – drives all business cycle fluctuations. As emphasized by [Ingram et al. \(1994\)](#), this one-shock assumption makes the real business cycle model stochastically singular: the model predicts that certain combinations of the endogenous variables will be deterministic. If, in the data, these exact linear relationships do not hold, any attempt to estimate (8) and (9) via maximum likelihood will fail.

One approach to coping with this stochastic singularity problem involves elaborating on the DSGE model by introducing additional structural disturbances – to preferences, technologies, monetary and fiscal policy rules, and so on – until the number of shocks equals the number of data series used in estimation. In fact, [Bencivenga \(1992\)](#), [Ireland \(1997, 2001a, b, 2002\)](#), [DeJong et al. \(2000a, b\)](#), [Kim \(2000\)](#), and [Schorfheide \(2000\)](#) all take exactly this approach. This strategy has its advantages: it serves to identify sources of aggregate fluctuations beyond the real business cycle model's technology shock and allows for a direct comparison of the relative importance of those additional disturbances in driving aggregate fluctuations. On the other hand, this strategy has its disadvantages, too: in particular, it requires the researcher to lean even harder on economic theory by making further, very specific, assumptions about the workings of the economy.

As an alternative approach to coping with the stochastic singularity problem, therefore, consider augmenting each equation in (9) with a serially correlated residual, or error term, so that the empirical model consists of (8),

$$\mathbf{f}_t = \mathbf{C}\mathbf{s}_t + \mathbf{u}_t \tag{10}$$

and

$$\mathbf{u}_t = \mathbf{D}\mathbf{u}_{t-1} + \boldsymbol{\xi}_t \tag{11}$$

for all $t = 0, 1, 2, \dots$, where the vector $\boldsymbol{\xi}_t$ of zero-mean, serially uncorrelated innovations is normally distributed with covariance matrix $E\boldsymbol{\xi}_t\boldsymbol{\xi}_t' = \mathbf{V}$ and is uncorrelated with the innovation ε_t to technology. This alternative approach – adding error terms to the observation equation (9) – is also used by [Altug \(1989\)](#), [McGrattan \(1994\)](#),

Hall (1996), and McGrattan et al. (1997) to estimate what would otherwise be stochastically singular real business cycle models. Each of these earlier studies follows Sargent (1989) by interpreting \mathbf{u}_t as a vector of measurement errors and by assuming that the matrices \mathbf{D} and \mathbf{V} are diagonal, so that the measurement errors are uncorrelated across variables. Here, by contrast, no such restrictions are imposed: the residuals in \mathbf{u}_t are allowed to follow a general, first-order vector autoregression. Thus, the residuals may still soak up measurement errors, but they can also be interpreted more liberally as capturing all of the movements and co-movements in the data that the real business cycle model, because of its elegance and simplicity, cannot explain. In this way, the hybrid model consisting of (8), (10), and (11) combines the power of the DSGE model with the flexibility of a VAR.

Conveniently, the hybrid model takes the form of a state-space econometric model; it can therefore be estimated via maximum likelihood, as described in Appendix B, once analogs to the model's variables Y_t , C_t , I_t , and H_t are found in the US data. Thus, in the data, C_t is defined as real personal consumption expenditures in chained 1996 dollars, investment is defined as real gross private domestic investment, also in chained 1996 dollars, and output Y_t is defined by the sum $C_t + I_t$. Hours worked H_t is defined as hours of wage and salary workers on private, non-farm payrolls. It is true that this measure of hours, which accounts for labor used to produce goods for export and for government purchase, does not correspond exactly to the measure of output $C_t + I_t$. Unfortunately, no easy way of correcting the hours data for this discrepancy exists; but extending the analysis using a more elaborate DSGE model that has implications for imports, exports, and government spending as well as for consumption and investment would certainly be a very useful task for future research.

Each series is converted to per-capita terms by dividing by the civilian, non-institutional population, age 16 and over. All data, except for population, are seasonally adjusted. Since the real business cycle model implies that output, consumption, and investment grow at the common rate η in steady state, the data are automatically detrended as part of the estimation process; they are not filtered in any other way. Data for consumption, investment, output, and population are taken from the Federal Reserve Bank of St. Louis' FRED database; data for hours worked come from the Bureau of Labor Statistics' Establishment Survey. The series are quarterly and run from 1948:1 through 2002:2.

The resource constraint (4) holds by construction in the data. Thus, only the series for Y_t , C_t , H_t are used in estimating the model; the series for I_t is redundant. For the purposes of estimation, therefore, \mathbf{f}_t , \mathbf{u}_t , and $\boldsymbol{\xi}_t$ reduce to 3×1 vectors, with

$$\mathbf{f}_t = [\ln(y_t/y) \quad \ln(c_t/c) \quad \ln(h_t/h)]',$$

$$\mathbf{u}_t = [u_{yt} \quad u_{ct} \quad u_{ht}]',$$

and

$$\boldsymbol{\xi}_t = [\xi_{yt} \quad \xi_{ct} \quad \xi_{ht}]'$$

for all $t = 0, 1, 2, \dots$, and the matrices \mathbf{D} and \mathbf{V} can be written as

$$\mathbf{D} = \begin{bmatrix} d_{yy} & d_{yc} & d_{yh} \\ d_{cy} & d_{cc} & d_{ch} \\ d_{hy} & d_{hc} & d_{hh} \end{bmatrix}$$

and

$$\mathbf{V} = \begin{bmatrix} v_y^2 & v_{yc} & v_{yh} \\ v_{yc} & v_c^2 & v_{ch} \\ v_{yh} & v_{ch} & v_h^2 \end{bmatrix}.$$

In estimating the hybrid model, the real business cycle model's structural parameters are constrained to satisfy the theoretical restrictions listed in Section 2, above. In addition, the eigenvalues of the matrix \mathbf{D} are constrained to lie inside the unit circle, so that the residuals in \mathbf{u}_t must be stationary. Finally, the covariance matrix \mathbf{V} is constrained to be positive definite. Again, Appendix B provides full details of the estimation procedure.

4. Results from the hybrid model

4.1. Parameter estimates

Preliminary attempts to estimate all of the hybrid model's parameters led, in particular, to an unreasonably low estimate of $\beta = 0.7941$ for the representative consumer's discount factor and an unreasonably high estimate of $\delta = 0.1810$ for the depreciation rate, given the quarterly time period. Within the real business cycle model, the low discount factor works to strengthen the consumer's preference for consumption today versus consumption tomorrow; similarly, the high depreciation rate works to make saving and investment less attractive. These estimates suggest, therefore, that the data might prefer a more elaborate version of the real business cycle model in which some agents appear to be less forward-looking than others due, perhaps, to myopia in preferences or borrowing constraints in financing investment. To be sure, this possibility deserves to be explored in more detail. Alternatively, data on interest rates and capital stocks, if added to the list of series used in the maximum likelihood procedure, might yield more satisfactory estimates of β and δ . Here, however, these extensions are left for future research; instead, the hybrid model is simply reestimated with β held fixed at 0.99 and δ held fixed at 0.025, the values originally suggested by Hansen (1985). Altug (1989) also finds it necessary to adopt this strategy of calibrating β and δ in order to successfully estimate the remaining parameters of a real business cycle model.

Accordingly, Table 1 reports maximum likelihood estimates of 21 parameters: the six structural parameters γ , θ , η , A , ρ , and σ from the real business cycle model, the nine elements of the matrix \mathbf{D} governing the persistence of the VAR residuals, and the six elements of the covariance matrix \mathbf{V} for the VAR residuals. The standard errors, also reported in Table 1, correspond to the square roots of the diagonal elements of minus one times the inverted matrix of second derivatives of the maximized log-likelihood

Table 1
Full sample estimates and standard errors

Parameter	Estimate	Standard error
γ	0.0045	0.0001
θ	0.2292	0.0065
η	1.0051	0.0005
A	5.1847	0.5048
ρ	0.9987	0.0018
σ	0.0056	0.0004
d_{yy}	1.3655	0.1572
d_{yc}	0.3898	0.1402
d_{yh}	-0.4930	0.1342
d_{cy}	0.1380	0.0712
d_{cc}	0.9690	0.0565
d_{ch}	-0.1046	0.0665
d_{hy}	0.7153	0.2123
d_{hc}	0.4605	0.1593
d_{hh}	0.2219	0.1566
v_y	0.0070	0.0013
v_c	0.0069	0.0007
v_h	0.0018	0.0021
v_{yc}	0.00002989	0.00001064
v_{yh}	0.00000903	0.00000722
v_{ch}	0.00001237	0.00000573

function. Calculating these standard errors requires two steps, numerically evaluating the matrix of second derivatives of the log-likelihood function and then inverting that very large matrix having elements of varying magnitudes, both of which may introduce approximation error into the statistics. Hence, these standard errors, though useful, do need to be interpreted with a bit of caution.

The point estimates of the real business cycle model's parameters, however, all appear quite reasonable. As discussed in the appendices, the estimates $A = 5.1847$ and $\gamma = 0.0045$ help match steady-state output, consumption, and hours worked in the model with the average levels of the same variables in the data. The estimate $\theta = 0.2292$ implies that capital's share in production is slightly less than 25%. The estimate $\eta = 1.0051$ makes the annualized, steady-state growth rate of real, per-capita output in the model equal to 2.06%. Finally, the estimate $\rho = 0.9987$ implies that the technology shock is extremely persistent, while the estimate $\sigma = 0.0056$ is of the same order of magnitude used throughout the literature on real business cycles.

The other estimates in Table 1 suggest, however, that there are important features of the data that the real business cycle model has difficulty explaining. The estimates imply, for instance, that the matrix \mathbf{D} has one real eigenvalue of modulus 0.9399 and two complex eigenvalues of modulus 0.8179; evidently, the residuals in \mathbf{u}_t are also quite persistent. Furthermore, the innovations in ξ_t have standard deviations of 0.0070, 0.0069, and 0.0018. Two of these three figures exceed the estimated standard deviation of the innovation ε_t to technology. On the other hand, the implied time series for ε_t

and the vector ξ_t , constructed using the Kalman smoothing algorithm alluded to in Appendix B, come quite close to being uncorrelated, consistent with the orthogonality assumption built into the hybrid model: calculations reveal that the correlation between ε_t and ξ_{yt} is -0.0634 , the correlation between ε_t and ξ_{ct} is 0.0133 , and the correlation between ε_t and ξ_{ht} is 0.0010 .

4.2. Explanatory power of the real business cycle model

What fraction of the observed output variation in the postwar US economy is explained by the real business cycle model? This question, first considered by Kydland and Prescott (1982), can also be addressed by using the estimated hybrid model to decompose the k -step-ahead forecast error variances in output, consumption, investment, and hours worked into two orthogonal components: one attributable to the real business cycle model's technology shock and the other attributable to the three residuals in \mathbf{u}_t . Table 2 displays the results of these forecast error variance decompositions.

In Table 2, the last line of panel A, with $k = \infty$, indicates that the technology shock accounts for nearly 90% of the unconditional variance in detrended output, but here, this result obtains despite the fact that the hybrid model also allows shocks to the elements of \mathbf{u}_t to help explain the behavior of output. Presumably, the residuals in \mathbf{u}_t pick up the combined effects of shocks, including monetary and fiscal policy shocks, not present in the real business cycle model. Here, therefore, Kydland and Prescott's (1982) finding appears to be robust to the inclusion of these additional shocks.

The robustness of Kydland and Prescott's (1982) finding can also be assessed, using the hybrid model estimated here, by attaching standard errors to each of the statistics reported in Table 2. Thus, standard errors also appear in the table, where they are computed by expressing each statistic as a function g of the vector Θ of estimated parameters and by calculating $[\partial g(\Theta)/\partial \Theta]' \mathbf{H} [\partial g(\Theta)/\partial \Theta]$, where \mathbf{H} is the covariance matrix of the estimated parameters in Θ and the derivatives $\partial g(\Theta)/\partial \Theta$ are evaluated numerically, as suggested by Runkle (1987).

The standard errors shown in Table 2 indicate that the statistical uncertainty surrounding the real business cycle model's ability to explain a substantial fraction of the observed output variation in the US data is large, though not as large as previously suggested by Eichenbaum (1991), who estimates the model's parameters via a method of moments procedure instead of the more efficient maximum likelihood technique used here. Even if the true fraction of output variation explained by the real business cycle model is two standard errors less than the point estimate of 90%, for instance, that fraction remains greater than 60%.

Other results displayed in Table 2 show that the technology shock accounts for more than 95% of the unconditional variance of detrended consumption and more than 50% of the unconditional variance of detrended investment, but almost none of the unconditional variance of hours worked. Thus, as noted by Cooley and Prescott (1995) among others, the real business cycle model does a much better job in explaining the behavior of output and its components than it does in explaining the behavior of hours worked.

Table 2
Forecast error variance decompositions

Quarters ahead	Percentage of variance due to technology	Standard error
(A) Output		
1	61.8430	10.5671
4	35.5003	6.9624
8	28.7467	6.7700
12	29.4831	7.5183
20	35.3378	9.1640
40	48.4763	11.0600
∞	89.9399	13.9401
(B) Consumption		
1	31.0978	6.2456
4	32.9700	6.7755
8	35.7260	8.1005
12	39.5799	9.2015
20	48.5522	10.8000
40	65.4138	11.3685
∞	95.7378	6.4410
(C) Investment		
1	44.0529	8.0988
4	25.2808	4.8840
8	18.2636	4.3737
12	17.4674	4.6860
20	18.8007	5.3565
40	21.6648	6.1571
∞	50.6782	32.5590
(D) Hours worked		
1	84.8526	31.3148
4	10.5126	2.1338
8	4.0181	0.9832
12	2.8049	0.8210
20	2.2471	0.7983
40	2.0734	0.8574
∞	2.0609	0.8770

As noted above, the technology shock accounts for almost 90% of the unconditional variance in aggregate output, and as shown in Table 2, it also accounts for more than 60% of the one-quarter-ahead forecast error variance in output. On the other hand, the technology shock accounts for less than half of the k -step-ahead forecast error variances for values of k ranging from 4 to 40, implying that the real business cycle model has more difficulty explaining output fluctuations over horizons between 1 and 10 years. These results are, of course, consistent with previous findings reported by Watson (1993), Cogley and Nason (1995), and Rotemberg and Woodford (1996); Watson (1993), in particular, finds that while the real business cycle model explains very high

and very low frequency movements in output, it is less successful in explaining those movements that take place at business cycle frequencies.

Finally, Table 2 contains a surprising result. Although, as noted above, technology shocks account for almost none of the unconditional variance of hours worked, they explain almost 85% of the one-quarter-ahead forecast error variance in the hours series. This result is encouraging, since it suggests that the real business cycle model still has some success in tracking quarter-to-quarter movements in aggregate employment, even if it fares less well in explaining movements over longer horizons.

4.3. Tests for parameter stability

One great strength of the real business cycle model is that it is supposed to be structural: it links the behavior of aggregate output and employment describing private agents' tastes and technologies – parameters that ought to remain constant, even across periods when monetary and fiscal policy regimes change. Here, the hybrid model can be used to test the hypothesis that these structural parameters do, in fact, appear stable over time. To check for parameter stability, the hybrid model is estimated over two disjoint subsamples: the first running from 1948:1 through 1979:4 and the second running from 1980:1 through 2002:2. The 1980 breakpoint corresponds, of course, to a date around which major changes in US monetary and fiscal policies are widely thought to have occurred.

Table 3 reports estimates of the hybrid model's parameters, along with their standard errors, for each of the two subsamples. For the six estimated parameters from the real business cycle model, differences do appear across the breakpoint. Specifically, the estimate of γ falls from 0.0046 before 1980 to 0.0042 after, probably in an attempt to explain the upward trend in per-capita hours worked that is examined in more detail by Motley (1997). The estimate of θ , meanwhile, rises from 0.2190 before 1980 to 0.2457 after, perhaps in an attempt to account for the 1990s investment boom. On the other hand, the estimate of η falls from 1.0053 before 1980 to 1.0046 after, and the estimate of A falls as well, suggesting that the productivity slowdown generated a downward shift in both the growth rate and the level of aggregate output. And while both estimates of ρ look quite similar to their full-sample counterpart, the estimates of σ pre and post-1980 are smaller than the full-sample estimate shown in Table 1. This last result – that aggregate shocks appear smaller when a one-time break in the process for output is allowed for – can also be found in work by Perron (1989) and Rappoport and Reichlin (1989).

Andrews and Fair (1988) describe procedures for testing whether differences like these are statistically significant. Let the vector Θ_q^1 contain any q parameters estimated with pre-1980 data, let Θ_q^2 contain the same q parameters estimated with post-1980 data, and let \mathbf{H}_q^1 and \mathbf{H}_q^2 denote the covariance matrices for Θ_q^1 and Θ_q^2 . Andrews and Fair (1988) show that the Wald statistic

$$W = (\Theta_q^1 - \Theta_q^2)'(\mathbf{H}_q^1 + \mathbf{H}_q^2)^{-1}(\Theta_q^1 - \Theta_q^2)$$

is asymptotically distributed as a chi-square random variable with q degrees of freedom under the null hypothesis of parameter stability: $\Theta_q^1 = \Theta_q^2$.

Table 3
Subsample estimates and standard errors

Parameter	Pre-1980 estimate	Standard error	Post-1980 estimate	Standard error
γ	0.0046	0.0001	0.0042	0.0002
θ	0.2190	0.0045	0.2457	0.0077
η	1.0053	0.0005	1.0046	0.0012
A	5.6309	0.2842	5.0535	0.9839
ρ	0.9935	0.0099	0.9966	0.0135
σ	0.0050	0.0015	0.0042	0.0009
d_{yy}	1.2553	0.2013	1.1390	0.1360
d_{yc}	0.1657	0.1558	0.6564	0.2970
d_{yh}	-0.3837	0.1633	-0.5332	0.1092
d_{cy}	0.1216	0.0889	0.0768	0.0999
d_{cc}	0.9065	0.0677	1.1172	0.1409
d_{ch}	-0.1143	0.0601	-0.1479	0.2176
d_{hy}	0.6380	0.2449	0.3722	0.1460
d_{hc}	0.1689	0.2225	0.5337	0.6014
d_{hh}	0.4232	0.2362	0.3562	0.1839
v_y	0.0090	0.0024	0.0056	0.0013
v_c	0.0084	0.0009	0.0051	0.0009
v_h	0.0049	0.0032	0.0013	0.0010
v_{yc}	0.00004982	0.00002080	0.00001542	0.00000774
v_{yh}	0.00002959	0.00003403	0.00000682	0.00000379
v_{ch}	0.00002201	0.00001458	0.00000518	0.00000595

Table 4
Tests for parameter stability

Stability of all 21 estimated parameters:	$W = 74.5383^{***}$
Stability of the 6 structural parameters:	$W = 15.3546^{**}$
Stability of the 15 remaining parameters:	$W = 54.1925^{***}$

Note: ** and *** indicate significance at the 5% and 1% levels.

Table 4 reports Wald statistics for the stability of all 21 estimated parameters from the hybrid model, the stability of the six structural parameters γ , θ , η , A , ρ , and σ from the real business cycle model, and the stability of the 15 remaining parameters from the matrices \mathbf{D} and \mathbf{V} governing the behavior of the VAR residuals. Across the board, the tests reject the null hypothesis of parameter stability. Evidently, important changes have taken place in the postwar US economy that neither the real business cycle model nor the hybrid model's residuals can fully account for. These test results echo and extend previous findings from Stock and Watson (1996), who record evidence of widespread instability in parameters from VAR models estimated with postwar US data.

4.4. Forecast accuracy

Table 5 reports on the accuracy of the hybrid model's out-of-sample forecasts. As noted above, the model has 21 estimated parameters. An unconstrained, first-order VAR model for the logs of per-capita output, consumption, and hours worked with a constant and a linear time trend for each variable also has 21 estimated parameters, making that VAR(1) a natural benchmark against which to judge the hybrid model's forecasting performance. Thus, the table compares the root-mean-squared forecast errors from the hybrid model to those from the VAR(1); for the sake of completeness, the table shows results from an unconstrained, second-order VAR model as well.

To create the statistics shown in Table 5, each of the models is estimated with data from 1948:1 through 1984:4 and used to generate out-of-sample forecasts one through four quarters ahead. Next, the sample is extended to 1985:1, and additional forecasts are generated using the updated estimates. Continuing in this way yields series of one-quarter-ahead forecasts running from 1985:1 through 2002:2, series of two-quarters-ahead forecasts running from 1985:2 through 2002:2, series of three-quarters-ahead forecasts running from 1985:3 through 2002:2, and series of four-quarters-ahead forecasts running from 1985:4 through 2002:2, all of which can be compared to the actual data that were realized over those periods.

The results indicate that in many cases – particularly for output and its components – forecasts from the hybrid model outperform those from the two unconstrained VAR models. To determine whether any of these differences are significant, Table 5 also reports a statistic that is used by Diebold and Mariano (1995) to test the null hypothesis of equal forecast accuracy across two models. Let $\{e_t^h\}_{t=1}^T$ denote a series of k -step-ahead forecast errors from the hybrid model, let $\{e_t^u\}_{t=1}^T$ denote the corresponding forecast errors from one of the unconstrained VAR models, and construct a sequence $\{l_t\}_{t=1}^T$ of loss differentials using $l_t = (e_t^u)^2 - (e_t^h)^2$ for all $t = 1, 2, \dots, T$. Diebold and Mariano (1995) show that the test statistic

$$S = l/\sigma_l$$

is asymptotically distributed as a standard normal random variable, where l is the sample mean of $\{l_t\}_{t=1}^T$ and where σ_l , the standard error of l , can be estimated according to the formulas given in their paper, under the null hypothesis of equal forecast accuracy: $l = 0$.

In Table 5, positive values of S indicate cases where the hybrid model's forecasts outperform the VAR model's, while negative values of S indicate cases where the opposite is true. In fact, tests of the null hypothesis $l = 0$ against the alternative $l > 0$ often reject the null of equal forecast accuracy. Meanwhile, only when the hybrid model's forecasts for hours worked are compared to those from the VAR(2) can the null of $l = 0$ be rejected in favor of the alternative that $l < 0$. Overall, therefore, the hybrid model's forecasting performance appears quite good.

As a final exercise, the hybrid model is reestimated while constraining the matrices **D** and **V** to be diagonal. This diagonal version of the hybrid model requires the real business cycle framework to account for all of the co-movements between the observed variables; as in the measurement-error models of Altug (1989), McGrattan (1994), Hall

Table 5
Forecast Accuracy, 1985:1–2002:2

Quarters Ahead	1	2	3	4
(A) Output				
RMSE: Hybrid	0.8319	1.6810	2.5706	3.4501
RMSE: Diagonal	0.6935	1.2201	1.7117	2.1613
RMSE: VAR(1)	1.0163	1.8602	2.6015	3.2472
RMSE: VAR(2)	0.9490	1.9783	2.9944	3.8845
S: Hybrid vs. VAR(1)	4.2047***	1.7496*	0.2110	−0.9795
S: Diagonal vs. VAR(1)	3.9323***	2.3712**	1.7474*	1.4071
S: Hybrid vs. VAR(2)	2.1937**	1.5608	1.1815	0.8642
S: Diagonal vs. VAR(2)	2.8708***	2.1497**	1.7631*	1.5245
(B) Consumption				
RMSE: Hybrid	0.5371	0.8849	1.2361	1.6208
RMSE: Diagonal	0.4781	0.7022	0.9093	1.1688
RMSE: VAR(1)	0.5554	0.8963	1.2110	1.5308
RMSE: VAR(2)	0.5854	1.0228	1.4828	1.9111
S: Hybrid vs. VAR(1)	0.9207	0.2572	−0.2818	−0.6440
S: Diagonal vs. VAR(1)	2.3321**	1.9189*	1.4721	1.1254
S: Hybrid vs. VAR(2)	1.1583	1.4070	1.2813	1.0203
S: Diagonal vs. VAR(2)	2.7518***	2.3575**	1.9652**	1.6083
(C) Investment				
RMSE: Hybrid	3.2441	5.8452	8.8561	11.9088
RMSE: Diagonal	3.1803	4.6807	6.1267	7.4359
RMSE: VAR(1)	4.0511	6.7590	9.1619	11.2436
RMSE: VAR(2)	3.4950	6.6361	9.9190	12.8333
S: Hybrid vs. VAR(1)	4.2287***	2.0408**	0.5264	−0.9300
S: Diagonal vs. VAR(1)	3.0864***	2.3256**	1.7907*	1.4927
S: Hybrid vs. VAR(2)	1.2140	1.3146	0.9559	0.6083
S: Diagonal vs. VAR(2)	1.0464	1.7969*	1.6306	1.4754
(D) Hours worked				
RMSE: Hybrid	0.5345	1.2973	2.2239	3.2436
RMSE: Diagonal	0.5663	1.0753	1.5589	2.0097
RMSE: VAR(1)	0.7439	1.4743	2.2072	2.9236
RMSE: VAR(2)	0.4458	1.1096	1.9253	2.7838
S: Hybrid vs. VAR(1)	4.3000***	1.5516	−0.0953	−1.1794
S: Diagonal vs. VAR(1)	3.5831***	2.5072**	2.2060**	2.0702**
S: Hybrid vs. VAR(2)	−4.4089***	−2.6781***	−2.1450**	−1.9447*
S: Diagonal vs. VAR(2)	−3.4761***	0.3892	1.7386*	1.8918*

Note: *, **, and *** indicate significance at the 10%, 5%, and 1% levels.

(1996), and McGrattan et al. (1997), the residuals are assumed to be uncorrelated. Table 6 displays the parameter estimates for this diagonal model, which differ in some cases from those shown for the more flexible specification in Table 1. Moreover, when

Table 6
Full sample estimates and standard errors: diagonal model

Parameter	Estimate	Standard error
γ	0.0038	0.0003
θ	0.5126	0.0760
η	1.0051	0.0011
A	0.9871	0.4904
ρ	0.9989	0.0015
σ	0.0091	0.0006
d_{yy}	-0.9843	0.0279
d_{cc}	0.9998	0.0003
d_{hh}	0.9942	0.0061
v_y	0.0001	0.0001
v_c	0.0061	0.0003
v_h	0.0073	0.0004

\mathbf{D} and \mathbf{V} are constrained to be diagonal, the maximized value of the log-likelihood function falls from 2323.6 to 2204.2: a likelihood ratio test easily rejects the null hypothesis that these constraints hold true. At the same time, however, Table 5 reveals that the diagonal version of the hybrid model often generates out-of-sample forecasts that outperform those from the more flexible hybrid model; by extension, the diagonal model's forecasts also outperform those from the two unconstrained VAR models.

5. Conclusion

This paper adds to a prototypical dynamic, stochastic, general equilibrium model – Hansen's (1985) real business cycle model with indivisible labor – a vector of residuals that follows a first-order autoregressive process. The result is a hybrid that exploits the power of detailed economic theory but remains flexible enough to be taken directly to the data: the model can be estimated via maximum likelihood and, once estimated, can be used to perform statistical hypothesis tests and to generate out-of-sample forecasts.

Some of the results presented above echo the well-known successes and shortcomings of the real business cycle framework as documented by Watson (1993), Cogley and Nason (1995), Cooley and Prescott (1995), and Rotemberg and Woodford (1996) among others. The results show, for example, that technology shocks do a better job in explaining the behavior of output and its components than they do in explaining the behavior of hours worked. In addition, technology shocks account for much of the variability in output that occurs over very short and very long horizons, but are less successful in accounting for output variation at business cycle frequencies. And finally, estimates of the model reveal that the statistical uncertainty surrounding Kydland and Prescott's (1982) finding that the real business cycle model explains a substantial fraction of the output variation in the US data is large, though not as large as suggested previously by Eichenbaum (1991).

Other results, however, illuminate aspects of the real business cycle model's performance that are less widely appreciated. For example, a statistical hypothesis test rejects the null hypothesis that the real business cycle model's structural parameters have remained stable over the postwar period. This result is disappointing, since it implies that the real business cycle model fails to live up to its promise of identifying parameters, describing private agents' tastes and technologies, that remain constant over time. On the other hand, the hybrid model developed here – which takes the real business cycle model as its starting point – delivers out-of-sample forecasts that frequently outperform those from an unconstrained VAR. This result is quite encouraging, since it indicates that the real business cycle model – often criticized for being an oversimplified abstraction – can actually help in answering the most practical of questions such as: what movements can we expect to see in aggregate output and employment in the US economy over the next quarter or the next year?

Finally, as the surveys in Cooley's (1995) volume make clear, work on dynamic, stochastic, general equilibrium theory has now moved well beyond its real business cycle origins to consider the effects of monetary and fiscal policy shocks, household production, imperfectly competitive market structures, and numerous other extensions. The method developed here can be applied more generally to take these extended models to the data and to assess their explanatory power, both within sample and out of sample. Doing so remains a task for future research.

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Appendix A. Solving the DSGE model

A.1. Equilibrium conditions

Eqs. (2)–(7) in the text describe the behavior of the model's six variables: Y_t , C_t , I_t , H_t , K_t , and A_t . These equations can be rewritten in terms of the six detrended variables

$y_t = Y_t/\eta^t$, $c_t = C_t/\eta^t$, $i_t = I_t/\eta^t$, $h_t = H_t$, $k_t = K_t/\eta^t$, and $a_t = A_t$ as

$$y_t = a_t k_t^\theta h_t^{1-\theta}, \quad (\text{A.1})$$

$$\ln(a_t) = (1 - \rho) \ln(A) + \rho \ln(a_{t-1}) + \varepsilon_t, \quad (\text{A.2})$$

$$y_t = c_t + i_t, \quad (\text{A.3})$$

$$\eta k_{t+1} = (1 - \delta)k_t + i_t, \quad (\text{A.4})$$

$$\gamma c_t h_t = (1 - \theta)y_t, \quad (\text{A.5})$$

and

$$\eta/c_t = \beta E_t\{(1/c_{t+1})[\theta(y_{t+1}/k_{t+1}) + 1 - \delta]\}. \quad (\text{A.6})$$

A.2. The steady state

In the absence of shocks, the economy converges to a steady state, in which each of the six stationary variables is constant, with $y_t = y$, $c_t = c$, $i_t = i$, $h_t = h$, $k_t = k$, and $a_t = a$ for all $t = 0, 1, 2, \dots$. Eq. (A.2) immediately provides the solution $a = A$.

Now suppose that the steady-state value y is in hand, and use (A.6) to solve for

$$k = \left(\frac{\theta}{\eta/\beta - 1 + \delta} \right) y.$$

Use (A.4) to solve for

$$i = \left[\frac{\theta(\eta - 1 + \delta)}{\eta/\beta - 1 + \delta} \right] y,$$

use (A.3) to solve for

$$c = \left\{ 1 - \left[\frac{\theta(\eta - 1 + \delta)}{\eta/\beta - 1 + \delta} \right] \right\} y,$$

and use (A.5) to solve for

$$h = \left(\frac{1 - \theta}{\gamma} \right) \left\{ 1 - \left[\frac{\theta(\eta - 1 + \delta)}{\eta/\beta - 1 + \delta} \right] \right\}^{-1}.$$

Finally, substitute these results back into (A.1) to solve for y :

$$y = a^{1/(1-\theta)} \left(\frac{\theta}{\eta/\beta - 1 + \delta} \right)^{\theta/(1-\theta)} \left(\frac{1 - \theta}{\gamma} \right) \left\{ 1 - \left[\frac{\theta(\eta - 1 + \delta)}{\eta/\beta - 1 + \delta} \right] \right\}^{-1}.$$

These equations show how the steady-state values y , c , i , h , k , and a depend on the parameters β , γ , θ , η , δ , and A . By contrast, the parameters ρ and σ have no impact on the model's steady state.

A.3. The linearized system

Eqs. (A.1)–(A.6) can be log-linearized to describe the behavior of the stationary variables as they fluctuate about their steady-state values in response to shocks. Let $\hat{y}_t = \ln(y_t/y)$, $\hat{c}_t = \ln(c_t/c)$, $\hat{i}_t = \ln(i_t/i)$, $\hat{h}_t = \ln(h_t/h)$, $\hat{k}_t = \ln(k_t/k)$, and $\hat{a}_t = \ln(a_t/a)$. Then first-order Taylor approximations to (A.1)–(A.6) yield

$$\hat{y}_t = \hat{a}_t + \theta \hat{k}_t + (1 - \theta) \hat{h}_t, \quad (\text{A.7})$$

$$\hat{a}_t = \rho \hat{a}_{t-1} + \varepsilon_t, \quad (\text{A.8})$$

$$(\eta/\beta - 1 + \delta) \hat{y}_t = [(\eta/\beta - 1 + \delta) - \theta(\eta - 1 + \delta)] \hat{c}_t + \theta(\eta - 1 + \delta) \hat{h}_t, \quad (\text{A.9})$$

$$\eta \hat{k}_{t+1} = (1 - \delta) \hat{k}_t + (\eta - 1 + \delta) \hat{h}_t, \quad (\text{A.10})$$

$$\hat{c}_t + \hat{h}_t = \hat{y}_t, \quad (\text{A.11})$$

and

$$0 = (\eta/\beta) \hat{c}_t - (\eta/\beta) E_t \hat{c}_{t+1} + (\eta/\beta + 1 - \delta) E_t \hat{y}_{t+1} - (\eta/\beta + 1 - \delta) \hat{k}_{t+1}. \quad (\text{A.12})$$

Eqs. (A.7)–(A.12) show that the model's dynamics depend on the model's parameters β , θ , η , δ , and ρ . By contrast, the parameters γ and A have no impact on the dynamics; they serve only to determine the steady state. And in this linearized system, of course, the standard deviation parameter σ determines the size of the technology shocks, but has no effect on the shapes of the impulse responses.

Eqs. (A.7)–(A.12) now form a system of linear stochastic difference equations. An application of Blanchard and Kahn's (1980) method yields a solution to this system—and hence an approximate solution to the real business cycle model—that takes the form of Eqs. (8) and (9) in the text.

Appendix B. Estimating the hybrid model

The hybrid model's likelihood function has as its arguments the real business cycle model's eight structural parameters, β , γ , θ , η , δ , A , ρ , and σ , the nine elements of the matrix \mathbf{D} governing the persistence of the residuals, and the six elements of the covariance matrix \mathbf{V} for the residuals. For any given set of values for these parameters, the procedure for evaluating the likelihood function begins by transforming the data series for output Y_t , consumption C_t , and hours worked H_t using the definitions

$$\hat{y}_t = \ln(Y_t) - t \ln(\eta) - \ln(y),$$

$$\hat{c}_t = \ln(C_t) - t \ln(\eta) - \ln(c),$$

and

$$\hat{h}_t = \ln(H_t) - \ln(h)$$

for all $t = 1, 2, \dots, T$, where T is the sample size. As noted in the text, these transformations work to detrend the data for output and consumption in a manner that is

consistent with the theoretical model: they assume that both variables grow at the common rate η in steady state. And as noted in Appendix A, the steady-state values y , c , and h are themselves functions of the structural parameters β , γ , θ , η , δ , and A . Thus, these data transformations imply that information on the average level and growth rate of each variable helps identify the parameters β , γ , θ , η , δ , and A .

Once these transformations are made,

$$\mathbf{f}_t = [\hat{y}_t \ \hat{c}_t \ \hat{h}_t]'$$

becomes a 3×1 vector of observables. Meanwhile, the 5×1 vector

$$\mathbf{x}_t = [\mathbf{s}'_t \ \mathbf{u}'_t]'$$

keeps track of the hybrid model's unobserved state variables: the capital stock \hat{k}_t and technology shock \hat{a}_t from the real business cycle model in \mathbf{s}_t and the VAR residuals in \mathbf{u}_t . Now the empirical model consisting of (8), (10), and (11) can be rewritten more compactly as

$$\mathbf{x}_t = \mathbf{F}\mathbf{x}_{t-1} + \mathbf{v}_t \tag{B.1}$$

and

$$\mathbf{f}_t = \mathbf{G}\mathbf{x}_t \tag{B.2}$$

for all $t = 1, 2, \dots, T$, where

$$\mathbf{F} = \begin{bmatrix} \mathbf{A} & \mathbf{0}_{2 \times 3} \\ \mathbf{0}_{3 \times 2} & \mathbf{D} \end{bmatrix},$$

$$\mathbf{G} = [\mathbf{C} \ \mathbf{I}_{3 \times 3}],$$

and $\mathbf{0}$ and \mathbf{I} denote zero and identity matrices with dimensions indicated by their subscripts. The serially uncorrelated innovation vector, constructed as

$$\mathbf{v}_t = [\mathbf{B}'\varepsilon_t \ \xi'_t]'$$

where ε_t is the innovation to the real business cycle model's technology shock and ξ_t contains the innovations to the VAR residuals, is normally distributed with mean zero and covariance matrix

$$E\mathbf{v}_t\mathbf{v}'_t = \mathbf{Q} = \begin{bmatrix} \sigma\mathbf{B}\mathbf{B}' & \mathbf{0}_{2 \times 3} \\ \mathbf{0}_{3 \times 2} & \mathbf{V} \end{bmatrix}.$$

Conveniently, (B.1) and (B.2) take the form of a state-space econometric model, allowing the procedure for evaluating the likelihood function to continue using the Kalman filtering algorithms outlined, for example, by Hamilton (1994, Chapter 13). Let

$$\hat{\mathbf{x}}_t = E(\mathbf{x}_t | \mathbf{f}_{t-1}, \mathbf{f}_{t-2}, \dots, \mathbf{f}_1)$$

denote the best estimate of the unobservable state vector \mathbf{x}_t for period t based on past observations of \mathbf{f}_t , and let

$$\Sigma_t = E(\mathbf{x}_t - \hat{\mathbf{x}}_t)(\mathbf{x}_t - \hat{\mathbf{x}}_t)'$$

denote the associated forecast error covariance matrix. Similarly, let

$$\hat{\mathbf{f}}_t = E(\mathbf{f}_t | \mathbf{f}_{t-1}, \mathbf{f}_{t-2}, \dots, \mathbf{f}_1)$$

denote the best forecast of \mathbf{f}_t based on past observations.

The Kalman filter takes the observations of \mathbf{f}_t for $t=1, 2, \dots, T$ as inputs, and works recursively to construct an implied series of forecast errors

$$\mathbf{w}_t = \mathbf{f}_t - \hat{\mathbf{f}}_t = \mathbf{f}_t - \mathbf{G}\hat{\mathbf{x}}_t$$

using the formulas

$$\hat{\mathbf{x}}_{t+1} = \mathbf{F}\hat{\mathbf{x}}_t + \mathbf{K}_t\mathbf{w}_t,$$

$$\mathbf{K}_t = \mathbf{F}\Sigma_t\mathbf{G}'(\mathbf{G}\Sigma_t\mathbf{G}')^{-1},$$

and

$$\Sigma_{t+1} = \mathbf{Q} + \mathbf{F}\Sigma_t\mathbf{F}' - \mathbf{F}\Sigma_t\mathbf{G}'(\mathbf{G}\Sigma_t\mathbf{G}')^{-1}\mathbf{G}\Sigma_t\mathbf{F}'$$

for all $t=1, 2, \dots, T$, together with the initial conditions $\hat{\mathbf{x}}_1$ and Σ_1 , derived from (B.1) and (B.2) as

$$\hat{\mathbf{x}}_1 = \mathbf{E}\mathbf{x}_1 = \mathbf{0}_{5 \times 1}$$

and

$$\text{vec}(\Sigma_1) = \text{vec}(\mathbf{E}\mathbf{x}_1\mathbf{x}_1') = (\mathbf{I}_{25 \times 25} - \mathbf{F} \otimes \mathbf{F})^{-1} \text{vec}(\mathbf{Q}).$$

Since, by construction, the forecast error \mathbf{w}_t is serially uncorrelated and normally distributed for all $t=1, 2, \dots, T$ with mean zero and covariance matrix

$$\mathbf{E}\mathbf{w}_t\mathbf{w}_t' = \mathbf{G}\Sigma_t\mathbf{G}',$$

the hybrid model's log-likelihood function can at last be calculated as

$$\ln L = -\frac{3T}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^T \ln |\mathbf{G}\Sigma_t\mathbf{G}'| - \frac{1}{2} \sum_{t=1}^T \mathbf{w}_t'(\mathbf{G}\Sigma_t\mathbf{G}')^{-1}\mathbf{w}_t.$$

The matrices \mathbf{A} , \mathbf{B} , and \mathbf{C} from (8) and (9), the approximate solution to the real business cycle model, enter into these Kalman filtering calculations. Since, as noted in the text and Appendix A, the elements of these matrices depend on the structural parameters β , θ , η , δ , and ρ , the dynamics in the data for output, consumption, and hours worked, together with the theoretical restrictions imposed by the real business cycle model, help identify β , θ , η , δ , and ρ . The matrices \mathbf{D} and \mathbf{V} , describing the persistence and volatility of the VAR residuals, and the parameter σ , describing the volatility of the real business cycle model's technology shock, also enter into the Kalman filtering calculations. Hence, the dynamics in the data also help identify the elements of \mathbf{D} and \mathbf{V} and the parameter σ .

Of course, once the hybrid model's likelihood function can be evaluated for any given set of parameter values, a numerical search algorithm can be employed to find the parameter values that maximize the likelihood function; these parameter values correspond to the maximum likelihood estimates reported in the text. In implementing this

maximum likelihood estimation procedure, additional constraints are imposed in various ways. For the real business cycle model's structural parameters, the numerical search algorithm is allowed to select values for the transformed parameters $\tilde{\gamma}$, $\tilde{\theta}$, $\tilde{\eta}$, \tilde{A} , and $\tilde{\sigma}$ that lie anywhere between positive and negative infinity; the original parameters are then calculated as $\gamma = |\tilde{\gamma}|$, $\theta = \tilde{\theta}^2 / (1 + \tilde{\theta}^2)$, $\eta = 1 + |\tilde{\eta}|$, $A = |\tilde{A}|$, and $\sigma = |\tilde{\sigma}|$ to insure that each satisfies the theoretical restrictions listed in Section 2. The parameter ρ , governing the persistence of the real business cycle model's technology shock, and eigenvalues of the matrix \mathbf{D} , governing the persistence of the VAR residuals, are constrained to be less than one in modulus by subtracting a very large number from the log likelihood function whenever one of these constraints is violated. Finally, the covariance matrix \mathbf{V} for the VAR residuals is constrained to be positive definite following the suggestion given by Hamilton (1994, p. 147), by calculating the Cholesky decomposition $\mathbf{V} = \tilde{\mathbf{V}}\tilde{\mathbf{V}}'$, where $\tilde{\mathbf{V}}$ is lower triangular, and allowing the numerical search algorithm to select values for the 6 non-zero elements of $\tilde{\mathbf{V}}$ that lie anywhere between positive and negative infinity; regardless, the implied matrix \mathbf{V} turns out to be symmetric and positive definite.

The Kalman filtering equations listed above can also be extended, using the calculations displayed by Hamilton (1994, pp. 394–397), to construct estimates of the innovations ε_t to technology and ξ_t to the VAR residuals based on the entire sample of data. These smoothed estimates are used above, in the text, to confirm that ε_t and ξ_t come close to being uncorrelated, consistent with the hybrid's model's assumptions.

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