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Implementing the Friedman rule

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Abstract

In cash-in-advance models, necessary and sufficient conditions for the existence of an equilibrium with zero nominal interest rates and Pareto optimal allocations place restrictions mainly on the very long-run, or asymptotic, behavior of the money supply. When these asymptotic conditions are satisfied, they leave the central bank with a great deal of flexibility to manage the money supply over any finite horizon. But what happens when these asymptotic conditions fail to hold? This paper shows that the central bank can still implement the Friedman rule if its actions are appropriately constrained in the short run.

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1. Introduction

Milton Friedman (1969) presents his famous rule for optimal monetary policymaking. "Our final rule for the optimum quantity of money," he writes (p. 34), "is that it will be attained by a rate of price deflation that makes the nominal rate of interest equal to zero." Friedman also suggests that this rule can be implemented by steadily contracting the money supply at the representative household's rate of time preference.

Wilson (1979) and Cole and Kocherlakota (1998) assess Friedman's proposals using fully-specified, general equilibrium models in which money is introduced through the imposition of a cash-in-advance constraint. These authors confirm the relevance of the Friedman rule by demonstrating that equilibrium allocations are efficient if and only

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if the nominal interest rate equals zero. However, they also find that the Friedman rule can be implemented through any one from a broad class of monetary policies. Some of these policies call for the money supply to expand over an arbitrarily long, but finite, horizon; others call for the money supply to contract, but at a rate that is slower than the representative household's rate of time preference. In fact, Wilson, and Cole and Kocherlakota show that necessary and sufficient conditions for the existence of an equilibrium with zero nominal interest rates and Pareto optimal allocations place restrictions mainly on the asymptotic behavior of the money supply: these restrictions require the money supply to eventually contract at a rate that is no faster than the representative household's rate of time preference.

These asymptotic conditions present a double-edged sword to a central banker who wishes to implement the Friedman rule. For when the conditions are satisfied, they leave the policymaker with considerable leeway in managing the money supply over any finite horizon. But what should a central banker do when, for reasons beyond his or her control, these asymptotic conditions fail to hold? Must the policymaker abandon the Friedman rule altogether? Or can he or she still find a way to manage the money supply so that nominal interest rates are zero and equilibrium allocations are efficient, at least in the short run?

To answer these questions, Section 2 sets up a cash-in-advance model like those used by Wilson (1979) and Cole and Kocherlakota (1998) and, for the sake of completeness, restates the asymptotic conditions that are both necessary and sufficient for implementing the Friedman rule over the infinite horizon. Section 3 then assumes that these asymptotic conditions do not hold and characterizes optimal monetary policies in this alternative case. There, the results indicate that the central bank can still implement the Friedman rule, but only if its policies are suitably constrained in the short run. Section 4 considers a stochastic variant of the same problem. Surprisingly, in this stochastic case, the conditions for implementing optimal allocations most closely resemble those derived for the original model in Section 2: once again, they serve mainly to restrict the long-run behavior of the money supply. Section 5 briefly concludes.

2. A cash-in-advance model

An infinitely-lived representative household has one unit of productive time during each period $t = 0, 1, 2, \dots$. Its preferences are described by the utility function

$$\sum_{t=0}^{\infty} \beta^t U(c_t, 1 - n_t),$$

where c_t denotes its consumption and $1 - n_t$ its leisure during period t. The discount factor satisfies $0 < \beta < 1$. The single-period utility function U is strictly increasing in both arguments, strictly concave, and twice continuously differentiable. Let U_i and U_{ij}, $i, j = 1, 2$, denote the first and second derivatives of U, and for $y \in (0, 1)$, define

$$V(y) = \frac{U_1(y, 1 - y)}{U_2(y, 1 - y)}.$$

It will be useful in all of what follows to assume that V is strictly decreasing with $\lim_{y\to 0} V(y) = \infty$ and $\lim_{y\to 1} V(y) = 0$. Since U is strictly increasing and strictly concave, a sufficient condition for $V' < 0$ is $U_{12} \geqslant 0$.

The household enters period t with money M_t and bonds B_t. The goods market opens first; here, the description of production and trade draws on Lucas (1980) interpretation of the cash-in-advance model. Suppose that the representative household consists of two members: a shopper and a worker. The shopper purchases consumption from workers from other households at the nominal price P_t, subject to the cash-in-advance constraint

$$\frac{M_t}{P_t} \geqslant c_t.$$

The worker, meanwhile, produces output according to the linear technology $y_t = n_t$ and sells this output to shoppers from other households for $P_t n_t$ units of money.

The asset market opens last. In this end-of-period asset market, the representative household receives a lump-sum nominal transfer H_t from the central bank and the household's bonds mature, providing B_t additional units of money. The household spends $B_{t+1}/(1+r_t)$ on new bonds, where r_t is the net nominal interest rate, and carries M_{t+1} units of money into period $t+1$. The household's budget constraint is, therefore,

$$\frac{M_t + H_t + B_t}{P_t} + n_t \geqslant c_t + \frac{B_{t+1}/(1+r_t) + M_{t+1}}{P_t}.$$

In addition to the cash-in-advance and budget constraints, the household's choices must satisfy the nonnegativity constraints

$$c_t \geqslant 0, \qquad 1 \geqslant n_t \geqslant 0, \qquad M_{t+1} \geqslant 0.$$

And while the household can choose negative values of B_{t+1}, it is not permitted to borrow more than it can ever repay. Let Q_t denote the present discounted value in the period 0 asset market of one unit of money received in the period t asset market, so that $Q_0 = 1$ and

$$Q_t = \prod_{s=0}^{t-1} \left(\frac{1}{1+r_s} \right) \quad \text{for } t = 1, 2, 3, \ldots.$$

Then the no-Ponzi-game constraints can be formalized as

$$W_{t+1} = Q_t \left(M_{t+1} + \frac{B_{t+1}}{1+r_t} \right) + \sum_{s=t+1}^{\infty} Q_s (H_s + P_s n_s) \geqslant 0.$$

Thus, the representative household chooses $\{c_t, n_t, M_{t+1}, B_{t+1}\}_{t=0}^{\infty}$ to maximize its utility function subject to its cash-in-advance, budget, nonnegativity, and no-Ponzi-game constraints, each of which must hold for all $t = 0, 1, 2, \ldots$. When the market-clearing conditions

$$y_t = c_t = n_t, \qquad M_{t+1} = M_t + H_t, \qquad B_{t+1} = 0$$

are imposed, necessary and sufficient conditions for a solution to the household's problem can be written as

$$U_1(y_t, 1-y_t) = \lambda_t + \mu_t, \tag{1}$$

$$U_2(y_t, 1-y_t) = \lambda_t, \tag{2}$$

$$\frac{\lambda_t}{P_t} = \frac{\beta(\lambda_{t+1} + \mu_{t+1})}{P_{t+1}}, \tag{3}$$

$$\frac{\lambda_t}{(1+r_t)P_t} = \frac{\beta\lambda_{t+1}}{P_{t+1}}, \tag{4}$$

$$\mu_t \geqslant 0, \qquad \frac{M_t}{P_t} \geqslant y_t, \qquad \mu_t\left(\frac{M_t}{P_t} - y_t\right) = 0, \tag{5}$$

for all $t = 0, 1, 2, \ldots,$ and

$$\lim_{t\to\infty} \frac{\beta^t \lambda_t M_{t+1}}{P_t} = 0, \tag{6}$$

where λ_t and μ_t are Lagrange multipliers on the budget and cash-in-advance constraints for period t. Accordingly, an equilibrium can be defined as a set of sequences $\{y_t, \lambda_t, \mu_t, r_t, P_t, M_{t+1}\}_{t=0}^{\infty}$ that satisfy (1)–(6), with the initial condition M_0 pinned down by a choice of nominal units.

Under the maintained assumptions on the household's utility function, there is a unique symmetric Pareto optimal allocation for this economy. This allocation has $y_t = y^*$ for all $t = 0, 1, 2, \ldots,$ where y^* satisfies the efficiency condition $V(y^*) = 1$: the marginal rate of substitution between leisure and consumption equals the corresponding marginal rate of transformation. What monetary policies, defined as sequences $\{M_{t+1}\}_{t=0}^{\infty}$, allow for the existence of an equilibrium in which allocations are Pareto optimal? To answer this question, Wilson (1979) and Cole and Kocherlakota (1998) present results like the following.

Proposition 1. *An equilibrium with $y_t = y^*$ for all $t = 0, 1, 2, \ldots$ exists if and only if*

$$\inf_t \beta^{-t} M_t > 0 \tag{7}$$

and

$$\lim_{t\to\infty} M_{t+1} = 0. \tag{8}$$

Proof. To begin, suppose that (7) and (8) are satisfied, and set $y_t = y^*$, $\mu_t = 0$, $r_t = 0$, $\lambda_t = U_1(y^*, 1-y^*) = U_2(y^*, 1-y^*)$, and $P_t = \beta^t P_0$ for all $t = 0, 1, 2, \ldots,$ where $P_0 > 0$ is chosen below. Clearly, these values satisfy (1)–(4). Since $\mu_t = 0$, (5) requires that

$$\beta^{-t} M_t \geqslant P_0 y^* \quad \text{for all } t = 0, 1, 2, \ldots.$$

But (7) guarantees the existence of $\varepsilon > 0$ such that $\beta^{-t} M_t \geqslant \varepsilon$ for all $t = 0, 1, 2, \ldots,$ and thereby allows this last condition to be satisfied for any choice of $P_0 \leqslant \varepsilon/y^*$. Meanwhile, (8) guarantees that (6) will hold. Thus, (7) and (8) are sufficient conditions for the existence of an optimal equilibrium.

Next, suppose that an equilibrium with $y_t = y^*$ for all $t = 0, 1, 2, \ldots$ exists. By (1)–(4), $\lambda_t = U_1(y^*, 1-y^*) = U_2(y^*, 1-y^*)$, $\mu_t = 0$, $r_t = 0$, and $P_t = \beta^t P_0 > 0$ for all $t = 0, 1, 2, \ldots$ in any such equilibrium. Thus, (5) requires that

$$\beta^{-t} M_t \geqslant P_0 y^* > 0 \quad \text{for all } t = 0, 1, 2, \ldots,$$

which implies that (7) must be satisfied. Meanwhile, (6) implies that (8) must hold. This establishes that (7) and (8) are also necessary conditions for the existence of an optimal equilibrium, completing the proof. □

Proposition 1 and its proof support Friedman's (1969) assertion that in monetary economies, Pareto optimal allocations are associated with price deflations and zero nominal interest rates. Friedman also suggests that his zero-nominal-interest-rate rule can be implemented by steadily contracting the money supply at the representative household's rate of time preference and, indeed, the policy that sets $M_t = \beta^t M_0$ for all $t = 0, 1, 2, \ldots$ satisfies both (7) and (8). As emphasized by Wilson (1979) and Cole and Kocherlakota (1998), however, many other monetary policies also satisfy (7) and (8), including ones that call for positive rates of money growth over arbitrarily long, but finite, horizons and ones that set $M_t = \pi^t M_0$, with $1 > \pi \geqslant \beta$, for all $t = 0, 1, 2, \ldots$.

In fact, although (7) does require the money supply to be strictly positive in every period, the additional constraints imposed by (7) and (8) apply only to the very long-run behavior of the money supply. Condition (7) places a lower bound on the asymptotic money growth rate: since the gross inflation rate equals β under the Friedman rule, the money stock must eventually grow at a rate that is at least as large as β to prevent the cash-in-advance constraint from binding. Condition (8) places an upper bound on the asymptotic money growth rate: evidently, the money supply must eventually contract to keep the nominal interest rate fixed at zero. Together, therefore, (7) and (8) simply require the money supply to asymptotically contract at a rate that is no faster than the representative household's rate of time preference.

3. Implementing the Friedman rule in the short run

When (7) and (8) hold, they leave the central bank with a great deal of flexibility; in fact, they allow the central bank to choose *any* time path for the money supply over *any* finite horizon while still implementing the Friedman rule. But what should a central banker do when (7) or (8) fails to hold?

When (7) fails to hold, the money supply contracts asymptotically at a rate that exceeds the representative household's rate of time preference. A second result, resembling those found in Woodford (1994), helps in considering this case.

Proposition 2. *Let the money supply contract at a constant rate that exceeds the representative household's rate of time preference, so that $M_{t+1}/M_t = \pi < \beta$ for all $t = 0, 1, 2, \ldots$. If the single-period utility function U takes the additively separable form*

$$U(c, 1-n) = u(c) + v(1-n), \tag{9}$$

where the functions u and v are strictly increasing, strictly concave, and twice continuously differentiable and if, in addition, the function u satisfies

$$\lim_{c \to 0} c\, u'(c) > 0, \tag{10}$$

then no equilibrium exists.

Proof. The proof proceeds in two steps. The first step shows that when $M_{t+1}/M_t = \pi < \beta$ for all $t = 0, 1, 2, \ldots$ and when utility is additively separable as in (9), the only equilibria that can possibly exist are those in which real balances approach zero asymptotically. The second step shows that under the additional assumption that (10) is satisfied, even those equilibria fail to exist.

Thus, to begin, suppose that $M_{t+1}/M_t = \pi < \beta$ for all $t = 0, 1, 2, \ldots$ and that utility is additively separable as in (9). Define the sequence $\{F_t\}_{t=0}^{\infty}$ by

$$F_t = \left(\frac{M_t}{P_t}\right) v'(1 - y_t) \geqslant 0 \quad \text{for all } t = 0, 1, 2, \ldots.$$

In any equilibrium, (2), (3), and (5) require that

$$\frac{v'(1 - y_t)}{P_t} = \frac{\lambda_t}{P_t} = \frac{\beta(\lambda_{t+1} + \mu_{t+1})}{P_{t+1}} \geqslant \frac{\beta\lambda_{t+1}}{P_{t+1}} = \frac{\beta v'(1 - y_{t+1})}{P_{t+1}},$$

and hence

$$F_t \geqslant \left(\frac{\beta}{\pi}\right) F_{t+1} > F_{t+1} \quad \text{for all } t = 0, 1, 2, \ldots.$$

Evidently, $\{F_t\}_{t=0}^{\infty}$ is strictly decreasing and bounded below by zero; it follows that this sequence converges to some number $F \geqslant 0$.

Next, define the sequence $\{G_t\}_{t=0}^{\infty}$ by

$$G_t = \left(\frac{M_t}{P_t}\right) u'(y_t) \geqslant 0 \quad \text{for all } t = 0, 1, 2, \ldots.$$

In any equilibrium, (1), (3), and (5) require that

$$\frac{u'(y_t)}{P_t} = \frac{\lambda_t + \mu_t}{P_t} \geqslant \frac{\lambda_t}{P_t} = \frac{\beta(\lambda_{t+1} + \mu_{t+1})}{P_{t+1}} = \frac{\beta u'(y_{t+1})}{P_{t+1}},$$

and hence

$$G_t \geqslant \left(\frac{\beta}{\pi}\right) G_{t+1} > G_{t+1} \quad \text{for all } t = 0, 1, 2, \ldots.$$

Thus, $\{G_t\}_{t=0}^{\infty}$ is also strictly decreasing and bounded below by zero; it, too, converges to some number $G \geqslant 0$.

Now define the sequence $\{D_t^1\}_{t=0}^{\infty}$ by

$$D_t^1 = M_t \left[\frac{\beta(\lambda_{t+1} + \mu_{t+1})}{P_{t+1}} - \frac{\lambda_t}{P_t}\right] \quad \text{for all } t = 0, 1, 2, \ldots.$$

Equation (3) and the assumptions that $M_{t+1}/M_t = \pi < \beta < 1$ imply that $D_t^1 = 0$ for all $t = 0, 1, 2, \ldots$ and that

$$\lim_{t \to \infty} D_t^1 = 0.$$

Meanwhile, (1) and (2) imply that

$$D_t^1 = \left(\frac{\beta}{\pi}\right) G_{t+1} - F_t.$$

Taken together, these last two results imply that

$$F = \left(\frac{\beta}{\pi}\right)G.$$

Finally, define the sequence $\{D_t^2\}_{t=0}^{\infty}$ by

$$D_t^2 = \left(\frac{\beta}{\pi}\right)G_t - F_t \quad \text{for all } t = 0, 1, 2, \ldots,$$

and note that under this definition,

$$\lim_{t\to\infty} D_t^2 = \left(\frac{\beta}{\pi}\right)G - F = 0.$$

Note also that in any equilibrium, (1), (2), and (5) require that

$$u'(y_t) = \lambda_t + \mu_t \geqslant \lambda_t = v'(1 - y_t) \quad \text{for all } t = 0, 1, 2, \ldots.$$

Since both u and v are strictly concave, this last set of requirements implies that $y_t \leqslant y^*$ for all $t = 0, 1, 2, \ldots$ where, as before, y^* is the unique value that satisfies $V(y^*) = 1$. More specifically, this last set of requirements implies that

$$\left(\frac{\beta}{\pi}\right)u'(y_t) - v'(1 - y_t) \geqslant \left(\frac{\beta}{\pi} - 1\right)u'(y_t) \geqslant \left(\frac{\beta}{\pi} - 1\right)u'(y^*) > 0$$

so that, in particular,

$$\left(\frac{\beta}{\pi}\right)u'(y_t) - v'(1 - y_t)$$

is bounded away from zero. The definitions

$$D_t^2 = \left(\frac{\beta}{\pi}\right)G_t - F_t = \left(\frac{M_t}{P_t}\right)\left[\left(\frac{\beta}{\pi}\right)u'(y_t) - v'(1 - y_t)\right]$$

then imply that for $\lim_{t\to\infty} D_t^2 = 0$ to hold as required,

$$\lim_{t\to\infty} \frac{M_t}{P_t} = 0$$

must also hold in any equilibrium. This establishes that when $M_{t+1}/M_t = \pi < \beta$ for all $t = 0, 1, 2, \ldots$ and when utility is additively separable as in (9), the only equilibria that can possibly exist are those in which real balances approach zero asymptotically.

To show that the additional restriction in (10) rules out all such equilibria, suppose to the contrary that an equilibrium of this type does exist when (10) is satisfied, and return to the definitions that imply

$$D_t^2 = \left(\frac{M_t}{P_t}\right)\left[\left(\frac{\beta}{\pi}\right)u'(y_t) - v'(1 - y_t)\right] \quad \text{for all } t = 0, 1, 2, \ldots.$$

Combine this last equality with (1), (2), and (5) to obtain

$$D_t^2 \geqslant \left(\frac{M_t}{P_t}\right)\left(\frac{\beta}{\pi} - 1\right)u'(y_t) \geqslant \left(\frac{\beta}{\pi} - 1\right)y_t u'(y_t) \geqslant 0 \quad \text{for all } t = 0, 1, 2, \ldots.$$

Since, as shown above, $\lim_{t\to\infty} D_t^2 = 0$ must hold in any equilibrium, it follows that

$$\lim_{t\to\infty} y_t u'(y_t) = 0$$

must hold as well. But $\lim_{t\to\infty} M_t/P_t = 0$ must also hold in any equilibrium; hence, (5) requires that

$$\lim_{t\to\infty} y_t = 0$$

as well. Taken together, however, these last two results contradict the assumption that (10) is satisfied. This establishes that when $M_{t+1}/M_t = \pi < \beta$ for all $t = 0, 1, 2, \ldots$ and when the utility function satisfies the restrictions in (9) and (10), no equilibrium exists. □

Scheinkman (1980), Lucas and Stokey (1987), and Woodford (1994) also use assumptions like (9) and (10). They interpret (10), in particular, as a condition that makes the gains from monetary trade sufficiently important to rule out equilibria in which self-fulfilling inflations drive the level of real balances to zero as the price level grows faster, or contracts more slowly, than the money supply. These conditions are satisfied by a wide range of utility functions, including those of the familiar form

$$U(c, 1-n) = \frac{c^{1-\sigma} - 1}{1-\sigma} + v(1-n) \quad \text{for all } \sigma \geqslant 1.$$

Proposition 2 suggests that when (7) fails to hold, the problem involves the likely nonexistence of an equilibrium, not just the suboptimality of equilibrium allocations. What happens when a central bank adopts a policy that is inconsistent with the existence of an equilibrium? Exploring the subtleties of this issue is left for future research; instead, the remainder of this paper focuses on the case in which the conditions of Proposition 1 are violated because (8) does not hold.

Suppose, for example, that a central banker is appointed at the beginning of period 0 and granted the authority to choose $\{H_t\}_{t=0}^{T-1}$, the monetary transfers for the first T periods. With the initial condition M_0 taken as given, this central banker's control over $\{H_t\}_{t=0}^{T-1}$ gives him or her control over $\{M_{t+1}\}_{t=0}^{T-1}$, the path for the money supply through the beginning of period T.

This central banker's term lasts for only T periods, however: during period T, a new central banker takes over and arbitrarily decides that the money supply will grow at the constant gross rate $\pi \geqslant 1$, so that $M_{T+j} = \pi^j M_T$ for all $j = 0, 1, 2, \ldots$. Under the maintained assumptions on the household's utility function, there is a unique steady-state equilibrium under this policy, in which output y_t and real balances $m_t = M_t/P_t$ are constant and equal to $\bar{y}$, where $\bar{y} < y^*$ uniquely satisfies $V(\bar{y}) = \pi/\beta$. So suppose in addition that, independent of the first central banker's decisions, $y_{T+j} = m_{T+j} = \bar{y}$ for all $j = 0, 1, 2, \ldots$.

The assumption that $\pi \geqslant 1$ implies that (8) will not hold when the first central banker takes office at the beginning of period 0. The question now becomes: Can this first central banker, through an appropriate choice of $\{M_{t+1}\}_{t=0}^{T-1}$, nevertheless guarantee the existence of an equilibrium in which nominal interest rates are zero and allocations are efficient, at least in the short run?

As a first step in answering this question, note that with $M_{T+j} = \pi^j M_T$ and $y_{T+j} = m_{T+j} = \bar{y}$ for all $j = 0, 1, 2, \ldots,$ (1)–(5) are satisfied with $\lambda_t = U_2(\bar{y}, 1-\bar{y})$, $\mu_t = U_1(\bar{y}, 1-\bar{y}) - U_2(\bar{y}, 1-\bar{y}) > 0$, $r_t = V(\bar{y}) - 1$, and $P_t = M_t/\bar{y}$ for all $t = T, T+1, T+2, \ldots,$ and (6) is satisfied as well. Hence, the values influenced by the first central banker, $\{y_t, \lambda_t, \mu_t, r_t, P_t, M_{t+1}\}_{t=0}^{T-1}$, need only satisfy (1), (2), and (5) for all $t = 0, 1, \ldots, T-1$, (3) and (4) for all $t = 0, 1, \ldots, T-2$,

$$\frac{\lambda_{T-1}}{P_{T-1}} = \frac{\beta U_1(\bar{y}, 1-\bar{y})\bar{y}}{M_T} \tag{11}$$

and

$$\frac{\lambda_{T-1}}{(1+r_{T-1})P_{T-1}} = \frac{\beta U_2(\bar{y}, 1-\bar{y})\bar{y}}{M_T}, \tag{12}$$

where these last two conditions correspond to (3) and (4) for $t = T-1$. These observations are useful in establishing the following result.

Proposition 3. *Suppose that $M_{T+j} = \pi^j M_T$, with $\pi \geqslant 1$, for all $j = 0, 1, 2, \ldots$ and that this policy puts the economy in its unique steady state from period T forward, with $y_{T+j} = m_{T+j} = \bar{y}$ for all $j = 0, 1, 2, \ldots$. Then an equilibrium with $y_t = y^*$ for all $t = 0, 1, \ldots, T-1$ exists if and only if*

$$M_T > 0 \tag{13}$$

and

$$M_t \geqslant \beta^t \left[\frac{U_1(y^*, 1-y^*)y^*}{\beta^T U_1(\bar{y}, 1-\bar{y})\bar{y}}\right] M_T \quad \textit{for all } t = 0, 1, \ldots, T-1. \tag{14}$$

Proof. To begin, suppose that (13) and (14) are satisfied, and set $\lambda_t = U_1(y^*, 1-y^*) = U_2(y^*, 1-y^*)$, $\mu_t = 0$, $y_t = y^*$, and

$$P_t = \beta^t \left[\frac{U_1(y^*, 1-y^*)}{\beta^T U_1(\bar{y}, 1-\bar{y})\bar{y}}\right] M_T,$$

for all $t = 0, 1, \ldots, T-1$. In addition, set $r_t = 0$ for all $t = 0, 1, \ldots, T-2$ and $r_{T-1} = V(\bar{y}) - 1$. Condition (13) guarantees that $P_t > 0$ for all $t = 0, 1, \ldots, T-1$, as required for the existence of this equilibrium. Clearly, (1) and (2) hold for all $t = 0, 1, \ldots, T-1$ and, since $P_{t+1} = \beta P_t$, (3) and (4) hold for all $t = 0, 1, \ldots, T-2$. Equations (11) and (12) hold as well. Since $\mu_t = 0$, (5) requires that

$$M_t \geqslant \beta^t \left[\frac{U_1(y^*, 1-y^*)y^*}{\beta^T U_1(\bar{y}, 1-\bar{y})\bar{y}}\right] M_T \quad \text{for all } t = 0, 1, \ldots, T-1,$$

but this condition coincides with (14) and is therefore guaranteed to hold. Thus, (13) and (14) are sufficient conditions for the existence of an equilibrium with $y_t = y^*$ for all $t = 0, 1, \ldots, T-1$.

Next, suppose that an equilibrium with $y_t = y^*$ for all $t = 0, 1, \ldots, T-1$ exists. By (1)–(3) and (11), $\lambda_t = U_1(y^*, 1-y^*) = U_2(y^*, 1-y^*)$, $\mu_t = 0$, and

$$P_t = \beta^t \left[\frac{U_1(y^*, 1-y^*)}{\beta^T U_1(\bar{y}, 1-\bar{y})\bar{y}}\right] M_T > 0,$$

for all $t = 0, 1, \ldots, T-1$ in any such equilibrium; this condition implies that (13) must hold. In addition, (5) requires that

$$M_t \geqslant \beta^t \left[\frac{U_1(y^*, 1-y^*)y^*}{\beta^T U_1(\bar{y}, 1-\bar{y})\bar{y}} \right] M_T \quad \text{for all } t = 0, 1, \ldots, T-1,$$

and this condition says that (14) must hold. This establishes that (13) and (14) are also necessary conditions for the existence of an equilibrium with $y_t = y^*$ for all $t = 0, 1, \ldots, T-1$, completing the proof. □

Before going on to interpret (13) and (14), it is useful to note that Proposition 3 holds much more generally. In particular, the assumption that the economy is in steady state from period T forward is not essential. All that is required is that the monetary policy adopted from period T forward give rise to an equilibrium in which the cash-in-advance constraint binds during period T, so that $M_T/P_T = y_T$ for some $y_T < y^*$. In the more general case, the proof goes through unchanged, with y_T in place of $\bar{y}$. As stated, however, the proposition makes clear that optimal allocations can be achieved in periods $t = 0, 1, \ldots, T-1$ even when the rate of money growth is positive, even when the cash-in-advance constraint binds, and even when allocations are suboptimal for *all* $t = T, T+1, T+2, \ldots$.

Proposition 3 indicates that the Friedman rule need not be abandoned when (8) fails to hold: the central bank can still select $\{M_{t+1}\}_{t=0}^{T-1}$ in a way that guarantees the existence of an equilibrium in which the nominal interest rate is zero for all $t = 0, 1, \ldots, T-2$ and allocations are efficient for all $t = 0, 1, \ldots, T-1$. Condition (13) simply insures that money is always in positive supply, given that (14) must hold for all $t = 0, 1, \ldots, T-1$ and that $M_{t+1} = \pi M_t$ for all $t = T, T+1, T+2, \ldots$. Condition (14), meanwhile, places upper and lower bounds on the money growth rate and thereby provides finite-horizon analogs to (7) and (8).

Consider (14) for $t = 0$. Since the initial condition M_0 is given, this constraint places an upper bound on M_T:

$$\left[\frac{\beta^T U_1(\bar{y}, 1-\bar{y})\bar{y}}{U_1(y^*, 1-y^*)y^*} \right] M_0 \geqslant M_T. \tag{15}$$

Thus, like (8), (14) implies that money growth must be sufficiently slow if the nominal interest rate is to remain at zero. Given a choice of M_T that satisfies (15), (14) also places lower bounds on M_t for all $t = 1, 2, \ldots, T-1$. Thus, like (7), (14) implies that money growth must be sufficiently fast to keep the cash-in-advance constraint from binding.

Conditions (13) and (14) still leave the central bank with some flexibility in choosing its policy: the money supply can expand for the first $T-1$ periods, for instance, provided that it contracts in period $T-1$ so that (15) holds. Unlike (7) and (8), however, (13) and (14) do impose nontrivial restrictions on the behavior of the money supply over a finite horizon. Thus, Proposition 3 requires the central bank to act in the short run to implement the Friedman rule in the short run.

4. Stochastic regime changes

As a variation on the same theme, suppose now that instead of taking place at the end of a finite horizon of fixed length, the regime change described above occurs randomly. More specifically, suppose that at the beginning of each period $t = 1, 2, 3, \ldots$, all agents observe a random signal that determines whether or not the first central banker's term will end. With probability $1 - \delta$, the first central banker continues in office, and with probability δ, the second central banker takes over during period t. These assumptions allow the first central banker to operate for sure during period 0; during each period that follows, however, there is a constant probability of the regime change. And once the regime change does occur, it cannot be reversed: the second central banker stays in office for the remainder of the infinite horizon.

As before, suppose that once in office, the second central banker arbitrarily decides to increase the money supply at the constant gross rate $\pi \geqslant 1$, placing the economy in its unique steady state. Thus, $M_{t+1}/M_t = \pi$ and $y_t = m_t = \bar{y}$ for all periods following the regime change, where $\bar{y}$ again satisfies $V(\bar{y}) = \pi/\beta$. Even for arbitrarily small values of $\delta > 0$, the stochastic process governing the timing of the regime change implies that the first central banker's term will be finite in length. The question remains: Can the first central banker, through an appropriate choice of policy, still guarantee the existence of an equilibrium in which allocations are efficient, with $y_t = y^*$, for all periods before the regime change?

To begin answering this question, note first that before the stochastic regime change, the representative household must make its decisions under uncertainty. Among the conditions that are both necessary and sufficient for a solution to the household's optimization problem, the intratemporal relationships (1), (2), and (5) remain as above. However, the intertemporal conditions (3), (4), and (6) generalize under uncertainty to

$$\frac{\lambda_t}{P_t} = \beta E_t\left(\frac{\lambda_{t+1} + \mu_{t+1}}{P_{t+1}}\right), \tag{16}$$

$$\frac{\lambda_t}{(1+r_t)P_t} = \beta E_t\left(\frac{\lambda_{t+1}}{P_{t+1}}\right), \tag{17}$$

and

$$\lim_{s\to\infty} E_t\left(\frac{\beta^s \lambda_s M_{s+1}}{P_s}\right) = 0, \tag{18}$$

where E_t denotes the household's rational expectation based on information available during period t. Each of these conditions must hold for all $t = 0, 1, 2, \ldots$.

Some additional notation will now prove useful. For all $t = 0, 1, 2, \ldots$, let y_t^1, λ_t^1, μ_t^1, r_t^1, P_t^1, and M_t^1 denote the values of y_t, λ_t, μ_t, r_t, P_t, and M_t that will prevail in equilibrium if the first central banker remains in power during period t. Similarly, for all $t = 1, 2, 3, \ldots$, let y_t^2, λ_t^2, μ_t^2, r_t^2, P_t^2, and M_t^2 denote the values that will prevail if the second central banker takes charge before or during period t. Since the economy reverts to its steady state after the regime change, $y_t^2 = \bar{y}$, $\lambda_t^2 = U_2(\bar{y}, 1-\bar{y}) = (\beta/\pi)U_1(\bar{y}, 1-\bar{y})$, $\mu_t^2 = U_1(\bar{y}, 1-\bar{y}) - U_2(\bar{y}, 1-\bar{y}) = (1-\beta/\pi)U_1(\bar{y}, 1-\bar{y}) > 0$, $r_t^2 = V(\bar{y}) - 1$,

$P_t^2 = M_t^2/\bar{y}$, and $M_{t+1}^2/M_t^2 = \pi \geqslant 1$ for all $t = 1, 2, 3, \ldots$. These values satisfy (1), (2), (5), and (16)–(18), as required, for all periods after the regime change.

For all periods before the regime change, (1), (2), (5), and (16)–(18) require that y_t^1, λ_t^1, μ_t^1, r_t^1, P_t^1, and M_t^1 satisfy

$$U_1\bigl(y_t^1, 1 - y_t^1\bigr) = \lambda_t^1 + \mu_t^1, \tag{19}$$

$$U_2\bigl(y_t^1, 1 - y_t^1\bigr) = \lambda_t^1, \tag{20}$$

$$\frac{\lambda_t^1}{P_t^1} = \beta(1-\delta)\left(\frac{\lambda_{t+1}^1 + \mu_{t+1}^1}{P_{t+1}^1}\right) + \beta\delta\left[\frac{U_1(\bar{y}, 1-\bar{y})\bar{y}}{M_{t+1}^1}\right], \tag{21}$$

$$\frac{\lambda_t^1}{(1+r_t^1)P_t^1} = \beta(1-\delta)\left(\frac{\lambda_{t+1}^1}{P_{t+1}^1}\right) + \beta\delta\left(\frac{\beta}{\pi}\right)\left[\frac{U_1(\bar{y}, 1-\bar{y})\bar{y}}{M_{t+1}^1}\right], \tag{22}$$

$$\mu_t^1 \geqslant 0, \qquad \frac{M_t^1}{P_t^1} \geqslant y_t^1, \qquad \mu_t^1\left(\frac{M_t^1}{P_t^1} - y_t^1\right) = 0, \tag{23}$$

and

$$\lim_{t\to\infty} \frac{[\beta(1-\delta)]^t \lambda_t^1 M_{t+1}^1}{P_t^1} = 0. \tag{24}$$

In deriving (21) and (22) from (16) and (17), use has been made of the fact that if the regime change occurs during period $t+1$, the beginning-of-period money supply M_{t+1} has still been determined by the past actions of the first central bank, so that $M_{t+1}^2 = M_{t+1}^1$.

In terms of this new notation, the first central banker takes the initial condition $M_0^1 = M_0$ as given and chooses an entire infinite sequence $\{M_{t+1}^1\}_{t=0}^{\infty}$, indicating how much money he or she plans to supply contingent on remaining in office during each period $t = 0, 1, 2, \ldots$. Of course, this infinite-horizon plan gets implemented only during periods before the stochastic regime change; however, the announcement of the entire sequence $\{M_{t+1}^1\}_{t=0}^{\infty}$ serves to pin down the representative household's expectations of what will happen so long as the first central banker remains in office. The key question from above can now be stated more precisely as: Can the first central banker, through an appropriate choice of $\{M_{t+1}^1\}_{t=0}^{\infty}$, guarantee the existence of a solution to (19)–(24) that has $y_t^1 = y^*$ for all $t = 0, 1, 2, \ldots$? The next result answers this question in the affirmative.

Proposition 4. *Suppose that at the beginning of each period $t = 0, 1, 2, \ldots$, there is a constant probability $\delta > 0$ that a new central banker will set $M_{t+j+1}/M_{t+j} = \pi \geqslant 1$ for all $j = 0, 1, 2, \ldots$ and that following this stochastic regime change, the economy reverts to its unique steady state, with $y_{t+j} = m_{t+j} = \bar{y}$ for all $j = 0, 1, 2, \ldots$. Then an equilibrium with $y_t = y_t^1 = y^*$ for all periods before the regime change exists if the sequence $\{M_{t+1}^1\}_{t=0}^{\infty}$ chosen by the first central banker satisfies*

$$M_{t+1}^1 > 0, \tag{25}$$

for all $t = 0, 1, 2, \ldots$,

$$\lim_{t\to\infty} \sum_{s=0}^{t-1} \bigl[\beta(1-\delta)\bigr]^s \left(\frac{1}{M_{s+1}^1}\right) < \infty, \tag{26}$$

and

$$\lim_{t\to\infty} M_{t+1}^1 = 0. \tag{27}$$

Proof. Suppose that (25)–(27) are satisfied, and set $y_t^1 = y^*$, $\lambda_t^1 = U_1(y^*, 1-y^*) = U_2(y^*, 1-y^*)$, and $\mu_t^1 = 0$ for all $t = 0, 1, 2, \ldots$. For $t = 1, 2, 3, \ldots$, set P_t^1 so that

$$\frac{1}{P_t^1} = \big[\beta(1-\delta)\big]^{-t}\left\{\frac{1}{P_0^1} - \beta\delta\left[\frac{U_1(\bar{y}, 1-\bar{y})\bar{y}}{U_1(y^*, 1-y^*)}\right]\sum_{s=0}^{t-1}\big[\beta(1-\delta)\big]^s\left(\frac{1}{M_{s+1}^1}\right)\right\},$$

where P_0^1 satisfying

$$\frac{1}{P_0^1} > \beta\delta\left[\frac{U_1(\bar{y}, 1-\bar{y})\bar{y}}{U_1(y^*, 1-y^*)}\right]\sum_{s=0}^{\infty}\big[\beta(1-\delta)\big]^s\left(\frac{1}{M_{s+1}^1}\right)$$

is chosen below. Condition (26), coupled with the fact that the sequence $\{X_t\}_{t=1}^{\infty}$ defined by

$$X_t = \sum_{s=0}^{t-1}\big[\beta(1-\delta)\big]^s\left(\frac{1}{M_{s+1}^1}\right)$$

for all $t = 0, 1, 2, \ldots$ is strictly increasing, guarantees that $P_t^1 > 0$ for all $t = 0, 1, 2, \ldots$, as required for the existence of this equilibrium. Finally, set

$$r_t^1 = \frac{\beta\delta\left(\frac{\pi-\beta}{\pi}\right)\left[\frac{U_1(\bar{y},1-\bar{y})\bar{y}}{M_{t+1}^1}\right]}{\beta(1-\delta)\left[\frac{U_1(y^*,1-y^*)}{P_{t+1}^1}\right] + \beta\delta\left(\frac{\beta}{\pi}\right)\left[\frac{U_1(\bar{y},1-\bar{y})\bar{y}}{M_{t+1}^1}\right]} > 0 \quad \text{for all } t = 0, 1, 2, \ldots.$$

Clearly, these values satisfy (19)–(22) for all $t = 0, 1, 2, \ldots$. Since $\mu_t^1 = 0$, (23) requires that

$$\big[\beta(1-\delta)\big]^{-t} M_t^1\left\{\frac{1}{P_0^1} - \beta\delta\left[\frac{U_1(\bar{y}, 1-\bar{y})\bar{y}}{U_1(y^*, 1-y^*)}\right]\sum_{s=0}^{t-1}\big[\beta(1-\delta)\big]^s\left(\frac{1}{M_{s+1}^1}\right)\right\} \geqslant y^*$$

for all $t = 1, 2, 3, \ldots$. But (26) implies that

$$\lim_{t\to\infty}\frac{[\beta(1-\delta)]^{t-1}}{M_t^1} = 0,$$

and together with (25), this last condition guarantees the existence of an $\varepsilon > 0$ such that $[\beta(1-\delta)]^{-t}M_t \geqslant \varepsilon$ for all $t = 0, 1, 2, \ldots$. For $t = 1, 2, 3, \ldots$, therefore, (23) is satisfied for any choice of P_0^1 such that

$$\frac{1}{P_0^1} \geqslant \frac{y^*}{\varepsilon} + \beta\delta\left[\frac{U_1(\bar{y}, 1-\bar{y})\bar{y}}{U_1(y^*, 1-y^*)}\right]\sum_{s=0}^{\infty}\big[\beta(1-\delta)\big]^s\left(\frac{1}{M_{s+1}^1}\right).$$

For $t = 0$, (23) requires that

$$\frac{M_0^1}{P_0^1} \geqslant y^*,$$

but since $M_0^1 \geqslant \varepsilon$, this condition holds as well. Finally, (26) and (27) guarantee that (24) will hold. Thus, as stated in the proposition, (25)–(27) are sufficient conditions for the existence of an equilibrium with $y_t = y_t^1 = y^*$ for all periods before the stochastic regime change. □

Condition (25) simply guarantees that money is always in positive supply. Conditions (26) and (27), meanwhile, bear a close resemblance to (7) and (8) from Proposition 1. Like (7), (26) places a lower bound on the asymptotic money growth rate: it requires that the money supply eventually grow at a rate that exceeds the probability-adjusted discount factor $\beta(1-\delta)$, ruling out policies under which the money supply contracts at so fast a rate that the representative household's demand for real balances becomes infinite. And like (8), (27) places an upper bound on the asymptotic money growth rate: once again, efficiency requires that the money supply eventually contract. Together, therefore, (26) and (27) only require the first central banker to promise that should he or she remain in office, the money supply will eventually contract at a rate that is no faster than the representative household's probability-adjusted discount rate $[\beta(1-\delta)]^{-1} - 1$.

Suppose, in particular, that the first central banker chooses a policy $\{M_{t+1}^1\}_{t=0}^{\infty}$ such that $M_{t+1}^1 = \gamma M_t^1$ for all $t = 0, 1, \ldots, T-1$ and $M_{t+1}^1 = \omega M_t^1$ for $t = T, T+1, T+2, \ldots$, with $\gamma > 1$, $\beta(1-\delta) < \omega < 1$, and $T < \infty$. Policies from this class, which call for the money supply to grow at a constant rate for the first T periods before contracting at a rate that is slower than the probability-adjusted discount rate thereafter, satisfy (25)–(27). By adopting one of these policies, therefore, the first central banker can guarantee the existence of an equilibrium in which allocations are Pareto optimal for all periods before the stochastic regime change. And if, after announcing a policy of this type, the first central banker loses power before the arrival of period T, the promised monetary contraction will never actually be observed! Strikingly, in this case, expectations of what the first central banker will do so long as he or she remains in office are by themselves sufficient to implement optimal allocations.

Surprisingly, therefore, the assumption that the regime change occurs randomly in any period, instead of for certain in a fixed period, actually weakens the constraints placed on a central banker who wishes to implement optimal allocations in the short run. The conditions imposed by Proposition 4, like those imposed by the original Proposition 1, serve mainly to restrict the long-run behavior of the money supply.

5. Conclusions

Wilson (1979) and Cole and Kocherlakota (1998) show that in cash-in-advance models, necessary and sufficient conditions for the existence of an equilibrium with zero nominal interest rates and Pareto optimal allocations place restrictions mainly on the asymptotic behavior of the money supply. For a central banker who wishes to implement the Friedman (1969) rule, these results are unambiguously positive. So long as the asymptotic conditions are guaranteed to hold, tremendous flexibility remains in how the money stock is managed over any finite horizon.

But what happens when these asymptotic conditions fail to hold? The two examples studied here indicate that the central bank can still find policies that implement optimal allocations, at least in the short run. In the first example, the asymptotic conditions fail to hold because the central banker remains in office for a fixed term of finite length. Nevertheless, by acting appropriately over his or her finite horizon, the central banker can still implement the Friedman rule. And in the second example, where the central banker's term ends in a randomly-determined period, the optimality conditions become even easier to satisfy. There, the central banker's leverage over expectations of what will happen in the distant future, so long as he or she remains in office, helps support optimal allocations even without direct action in the short run. These results, too, provide unambiguously good news for central bankers who wish to implement the Friedman rule.

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