A Reconsideration of Money Growth Rules*

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Abstract

A New Keynesian model, estimated using Bayesian methods over a sample period that includes the 2009-15 episode of zero nominal interest rates, illustrates the effects of replacing the Federal Reserve's historical policy of interest rate management with one targeting money growth instead. Counterfactual simulations show that a rule for adjusting the money growth rate, modestly and gradually, in response to changes in the output gap delivers performance comparable to the estimated interest rate rule in stabilizing output and inflation. The simulations also reveal that, under the same money growth rule, the US economy would have recovered more quickly from the 2007-9 recession, with a much shorter period of exceptionally low interest rates. These results suggest that money growth rules can serve as simple but useful guides for monetary policy and eliminate concerns about monetary policy effectiveness when the zero lower bound constraint is binding.

JEL Codes: E31, E32, E41, E47, E51, E52.

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1 Introduction

For the past quarter century, and perhaps longer, the Federal Reserve has conducted monetary policy by managing nominal interest rates. While today's practice of strict federal funds rate targeting has its origins in the early 1990s, Greenspan (1997), Meulendyke (1998), and Thornton (2006) all describe Federal Reserve policy as shifting towards tighter interest rate control beginning sometime in the 1980s. Cook (1989) and Gilbert (1994) go back even further, arguing that the reserves targeting procedures used from 1979 through 1982 disguised policy actions taken to manage the funds rate instead.

Academic economists also depict Federal Reserve policy as managing interest rates. Taylor (1993) introduces his now-famous rule, which describes how the Fed adjusts its interest rate target in response to movements in the output gap and inflation. Taylor (1993) also demonstrates that the strikingly simple formula tracks actual movements in the federal funds rate remarkably well over the period from 1987 through 1992. Some variant of the Taylor rule now appears as the description of monetary policy in textbook New Keynesian models presented, for example, by Woodford (2003) and Galí (2015).

Preference for interest rate management, in both practice and theory, often is motivated with reference to Poole's (1970) classic analysis, demonstrating that in a stochastic IS-LM model, policies targeting the nominal interest rate insulate output from the effects of money demand shocks, whereas policies targeting the money stock instead allow these shocks to contribute to macroeconomic volatility. Poole's model holds the aggregate price level fixed, but Ireland (2000), Collard and Dellas (2005), and Galí (2015) demonstrate that these results extend to modern New Keynesian models as well, in which monetary policies calling for a constant rate of money growth lead to excess volatility in both output and inflation, compared to policies targeting interest rates instead, especially when the economy is hit by recurrent money demand shocks. Furthermore, as emphasized by Ireland (2004*c*) and Belongia and Ireland (2021), standard New Keynesian models feature forward-looking variants of more traditional Keynesian IS and Phillips curves that imply monetary policy affects output and inflation exclusively through its ability to influence the current and expected future path for the short-term nominal interest rate. The Taylor rule, therefore, becomes a natural benchmark for describing monetary policy in these models. And, to the extent that Federal Reserve officials also believe that monetary policy influences economic activity mainly if not entirely through the New Keynesian interest rate channel, it makes sense for them to focus on managing interest rates as well.

Recent events, however, prompt a reconsideration of the prevailing consensus favoring interest rate rules. First and most obviously, the extended period from 2009 through 2015, during which the Federal Reserve's traditional federal funds rate targeting procedures were constrained by the perceived lower bound on nominal interest rates, raises the question of whether alternative policy rules focused on managing the money stock might have allowed the Fed to pursue its stabilization objectives more effectively during and after the financial crisis and Great Recession of 2007-9. Belongia and Ireland (2017, 2018) and Keating, Kelly, Smith, and Valcarcel (2019) present empirical evidence suggesting this may have been the case, but stop sort of exploring the possibility, theoretically, in the standard New Keynesian framework. Second, the Fed's actual response to continued weakness in output and inflation while the funds rate remained in a target range near zero went beyond three waves of large-scale asset purchases. Also required were other important changes in operating procedures, such as the introduction of interest payments on bank reserves and the establishment of a reverse repurchase agreement program though which the Fed interacted with a wide range of nonbank financial institutions. Although moving from a policy involving interest rate management to one of targeting the growth rate of a monetary aggregate might once have seemed prohibitively difficult, this recent experience shows, to the contrary, that operating procedures and institutional arrangements can be changed significantly, even on short notice, to support any major shift in policy regime. Finally, while the previous studies by Ireland (2000), Collard and Dellas (2005), and Galí (2015) all suggest that policy rules calling for *constant* rates of money growth will perform poorly, relative to Taylor rules, in stabilizing output and inflation, none of these studies considers the possibility that money growth rules might work significantly better if they allowed policy to adjust to movements in the output gap and inflation in a manner similar to that of the Taylor rule.

Thus, this paper extends previous work by reconsidering money growth rules in an estimated New Keynesian model. By identifying a parsimonious rule that dictates a systematic response of money growth to changes in the output gap, it follows in the same style of research presented, for instance, in Taylor (1999) by characterizing rules that remain simple while still delivering favorable economic outcomes. And by using counterfactual simulations to assess how the US economy would have performed over a sample period running from 1983 through 2019, it illustrates the satisfactory performance of a money growth rule in both good times – the period of the Great Moderation – and bad – the Great Recession and its aftermath.[1]

The particular variant of the New Keynesian model used here takes most of its basic features from those in Ireland (2004*b*, 2004*c*, 2007, 2011), but innovates in four distinct ways. First, it introduces real money balances into a representative household's utility function in a manner that leaves the New Keynesian IS and Phillips curves in their standard forms, excluding the additional terms involving money growth that appear in Ireland (2004*c*). This ensures that the extended model retains the New Keynesian assumption that monetary policy actions have an impact on output only through their effects on the current and expected future path of the short-term nominal interest rate. The intent is to put money growth rules to a most stringent test, by excluding model features that might specifically favor stability in the money stock.

Second, the model's money-in-the-utility function specification is also tailored to imply that the level of real balances demanded by the non-bank public remains finite even as nominal interest rates fall to zero, reflecting observations made by Ireland (2009) and Rognlie

[1]Billi, Söderström, and Walsh (2021) extend previous results from Ireland (2000), Collard and Dellas (2005), and Galí (2015) in another way, by using a calibrated New Keynesian model to show that interest rate rules targeting a constant money growth rate can outperform those targeting inflation – though not the price level – once the zero lower bound is accounted for.

(2016) that US money demand did not explode during either episode of very low nominal interest rates following the 2001 and 2007-9 recessions. Intriguingly, as noted by Rognlie (2016), this specification implies that short-term interest rates *can* fall below zero, at least by modest amounts for short periods of time – a phenomenon that will be explored in the counterfactual experiments performed with the estimated model.[2] Third, the model includes adjustment costs of real balances in its specification, following Nelson (2002) and Andrés, López-Salido, and Nelson (2004, 2009), all of which present evidence that New Keynesian models with money fit the data better when they allow for gradual adjustment of real balances to shocks that hit the economy.

Fourth and finally, the analysis here employs methods developed by Kulish, Morley, and Robinson (2017) to account for periods, like that experienced in the US from 2009 through 2015, when short-term nominal interest rates were constrained by the Fed to remain near zero. According to the New Keynesian model, even after its current policy rate is lowered to zero, the central bank can use "forward guidance," in the form of policy announcements that lengthen private agents' expectations regarding the duration of the zero interest rate episode, to deliver additional monetary stimulus. The Bayesian estimation methods used here exploit survey data to track changes in the expected duration of the zero interest rate period and the effects these shifts in expectations have on output and inflation. Thus, with these methods, the model can be estimated over a sample running continuously from 1983 through 2019, accounting for the effects of both zero interest rates and forward guidance over the 2009-15 period as well as the effects of more traditional interest rate policy before and after. The estimated model can then be used to explore counterfactual scenarios in which the central bank systematically adjusts its target for the money growth rate under both favorable and unfavorable economic conditions.

[2] Recently, negative policy rates have been implemented by interest rate targeting central banks in Denmark, the Euro Area, Japan, Sweden, and Switzerland. The Swiss National Bank, in particular, has held its key policy rate at −0.75 percent since 2015. See Jackson (2015) for a discussion of the international experience with negative policy rates and Abo-Zaid and Garín (2016) and Dong and Wen (2017) for other theoretical models that allow for negative nominal interest rates.

The results from this exercise reveal that, even in a model that departs minimally from standard New Keynesian specifications and therefore offers no special role for changes in the money stock, a money growth rule nonetheless can deliver performance on par with that generated by more conventional Taylor rules for the interest rate. The counterfactual simulations show, in particular, that under a money growth rule that responds modestly but persistently to changes in the output gap, the US economy would have recovered more quickly than it actually did from the financial crisis and Great Recession, without requiring a prolonged period of zero or negative interest rates. Thus, the results suggest that as Federal Reserve officials search for a new policy framework within which they can more reliably achieve their stabilization objectives in an environment with low interest rates and inflation following a series of adverse disturbances, abandoning the traditional practice of managing the federal funds rate in favor of a rule targeting the money growth rate should be added to the list of possibilities considered.

2 The Model

2.1 Overview

The model economy consists of a representative household, a representative finished goods-producing firm, a continuum of intermediate goods-producing firms indexed by $i \in [0,1]$, and a central bank. During each period $t = 0, 1, 2, \ldots$, each intermediate goods-producing firm produces a distinct intermediate good. Hence, intermediate goods are also indexed by $i \in [0,1]$, with good i produced by firm i. The model features enough symmetry, however, to allow the analysis to focus on the behavior of a representative intermediate goods-producing firm that manufactures the generic intermediate good i.

Habit formation introduced through the representative household's preferences and incomplete indexation of sticky nominal goods prices set by monopolistically competitive intermediate goods-producing firms imply that the model's New Keynesian IS and Phillips curves include both backward and forward-looking elements. The estimation procedure allows the data to decide on the relative importance of these backward and forward-looking terms. The central bank in the estimated model is assumed to conduct monetary policy according to a version of the Taylor (1993) rule, the standard New Keynesian representation of the Federal Reserve's actual practice of federal funds rate targeting over most of the 1983-2019 sample period, with a switch to forward guidance during the zero interest rate episode of 2009-15. As noted above, however, the introduction of a money demand curve of a form that is consistent with the same US data also permits consideration of counterfactual monetary policy rules for money growth targeting instead.

2.2 The Representative Household

The representative household enters each period $t = 0, 1, 2, \ldots$ with M_{t-1} units of money and B_{t-1} bonds. At the beginning of period t, the household receives a lump-sum monetary transfer T_t from the central bank. In addition, the household's bonds mature, yielding B_{t-1} additional units of money. The household uses some of this money to purchase B_t new bonds at the price of $1/r_t$ units of money per bond; thus, r_t denotes the gross nominal interest rate between t and $t+1$.

During period t, the household supplies $h_t(i)$ units of labor to each intermediate goods-producing firm $i \in [0, 1]$. The household gets paid at the nominal wage rate W_t, earning $W_t h_t$ in labor income, where h_t denotes total hours worked during the period. Also during period t, the household consumes C_t units of the finished good, purchased at the nominal price P_t from the representative finished goods-producing firm.

At the end of period t, the household receives nominal profits $D_t(i)$ from each intermediate goods-producing firm $i \in [0, 1]$. The household then carries M_t units of money into period $t+1$, chosen subject to the budget constraint

$$\frac{M_{t-1} + T_t + B_{t-1} + W_t h_t + D_t}{P_t} \geq C_t + \frac{M_t + B_t/r_t}{P_t} \tag{1}$$

for all $t = 0, 1, 2, \ldots$, where D_t denotes total profits received for the period.

The household's preferences are described by the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t a_t \left[\ln(C_t - \gamma C_{t-1}) + v\left(\frac{M_t}{P_t Z_t}, u_t\right) - \frac{\phi_m}{2}\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right)^2 \left(\frac{M_t}{P_t Z_t}\right) - \frac{\chi_0 h_t^{1+\chi}}{1+\chi} \right] \tag{2}$$

where both the discount factor and the habit formation parameter lie between zero and one, with $0 < \beta < 1$ and $0 \leq \gamma \leq 1$, the weight on the disutility of labor is strictly positive, with $\chi_0 > 0$, and the inverse of the Frisch elasticity of labor supply is nonnegative, with $\chi \geq 0$. The preference shock a_t follows the stationary autoregressive process

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at} \tag{3}$$

for all $t = 0, 1, 2, \ldots$, with $0 \leq \rho_a < 1$, where the serially uncorrelated innovation ε_{at} is normally distributed with mean zero and standard deviation σ_a. Utility is additively separable across consumption, real balances, and hours worked so as to imply a specification for the model's IS curve that does not include terms involving money and employment. As shown by King, Plosser, and Rebelo (1988), additive separability also implies that a logarithmic specification over consumption is needed for the model to be consistent with balanced growth. Also for balanced growth, real balances M_t/P_t enter utility through the function v after being scaled by the aggregate productivity shock Z_t, which follows a random walk with drift:

$$\ln(Z_t) = \ln(z) + \ln(Z_{t-1}) + \varepsilon_{zt} \tag{4}$$

for all $t = 0, 1, 2, \ldots$, with $z > 1$, where the serially uncorrelated innovation ε_{zt} is normally distributed with mean zero and standard deviation σ_z. The shock u_t to money demand follows the stationary autoregressive process

$$\ln(u_t) = \rho_u \ln(u_{t-1}) + \varepsilon_{ut} \tag{5}$$

for all $t = 0, 1, 2, \ldots$, with $0 \leq \rho_u < 1$, where the serially uncorrelated innovation ε_{ut} is normally distributed with mean zero and standard deviation σ_u.

Finally, the parameter $\phi_m \geq 0$ governs the magnitude of the quadratic adjustment cost for real balances. Since these costs subtract from utility along with hours worked, they have the interpretation of a time cost, and are scaled by the average growth rate parameter z from (3) so as to equal zero in the model's steady state. Chow (1966) argues that adjustment cost specifications analogous to those used in models of capital and durable consumer goods accumulation – and by extension similar to those used here – are needed to reconcile empirical models of long and short-run money demand. Nelson (2002) introduces them explicitly into a New Keynesian model, with reference to Goldfeld's (1973) partial-adjustment equation for money demand. Andrés, López-Salido, and Nelson (2004, 2009) use maximum likelihood methods to estimate statistically significant adjustment cost parameters for real balances in two variants of the New Keynesian model. Here, in a similar spirit, adjustment costs for real balances are included in the theoretical model so as to allow the Bayesian estimation procedure to reveal information about their importance via the location and shape of the posterior distribution for the parameter ϕ_m relative to a prior distribution that is chosen, as described below, to allow for values close to zero.

Thus, the household chooses C_t, h_t, B_t, and M_t for all $t = 0, 1, 2, \ldots$ to maximize expected utility (2) subject to the budget constraint (1) for all $t = 0, 1, 2, \ldots$. Section 1 of the appendix lists the first-order conditions for this problem and shows, as well, that in the special case where $\gamma = 0$ and $\phi_m = 0$, so that there is no habit formation in consumption or adjustment costs for real balances, these first-order conditions can be combined to obtain a money demand curve defined implicitly by

$$v_1\left(\frac{M_t}{P_t Z_t}, u_t\right) = \frac{Z_t}{C_t}\left(1 - \frac{1}{r_t}\right), \tag{6}$$

where v_1 denotes the partial derivative of the function v with respect to its first argument, scaled real balances.

Adapting a specification suggested by Rognlie (2016), suppose that the utility function over real balances is such that

$$v_1\left(\frac{M_t}{P_t Z_t}, u_t\right) = \frac{1}{\delta}\left[\ln(m^*) - \ln\left(\frac{M_t}{P_t Z_t}\right) + \ln(u_t)\right], \tag{7}$$

where $\delta > 0$ and m^* is a satiation level of scaled real balances, beyond which additional money holdings begin to impose marginal costs on, instead of yielding marginal benefits to, the household. Then (6) specializes to

$$\ln\left(\frac{M_t}{P_t Z_t}\right) = \ln(m^*) - \delta\left(\frac{Z_t}{C_t}\right)\left(1 - \frac{1}{r_t}\right) + \ln(u_t) \tag{8}$$

confirming that, along a steady-state growth path, real balances and consumption grow at the same rate z. Meanwhile, looking across steady-state growth paths, δ governs the interest semi-elasticity of money demand. Intuitively, when habit formation in consumption and adjustment costs of real balances reappear in the more general model that is estimated below, real balances and consumption gradually adjust towards their long-run desired levels satisfying (8), much as they do in Chow's (1966) original analysis.

2.3 The Representative Finished Goods-Producing Firm

During each period $t = 0, 1, 2, \ldots$, the representative finished goods-producing firm uses $Y_t(i)$ units of each intermediate good $i \in [0, 1]$, purchased at the nominal price $P_t(i)$, to manufacture Y_t units of the finished good according to the technology described by

$$\left[\int_0^1 Y_t(i)^{(\theta_t-1)/\theta_t}\,\mathrm{d}i\right]^{\theta_t/(\theta_t-1)} \geq Y_t,$$

where θ_t translates into a random shock to the intermediate goods-producing firms' desired markup of price over marginal cost and therefore acts like a cost-push shock of the kind introduced into the New Keynesian model by Clarida, Galí, and Gertler (1999). Here, this markup shock follows the stationary autoregressive process

$$\ln(\theta_t) = (1-\rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \varepsilon_{\theta t} \tag{9}$$

for all $t = 0, 1, 2, \ldots$, with $0 \leq \rho_\theta < 1$ and $\theta > 1$, where the serially uncorrelated innovation $\varepsilon_{\theta t}$ is normally distributed with mean zero and standard deviation σ_θ. Section 1 of the appendix lists the conditions summarizing the finished goods-producing firm's profit-maximizing behavior.

2.4 The Representative Intermediate Goods-Producing Firm

During each period $t = 0, 1, 2, \ldots$, the representative intermediate goods-producing firm hires $h_t(i)$ units of labor from the representative household to manufacture $Y_t(i)$ units of intermediate good i according to the technology described by

$$Z_t h_t(i) \geq Y_t(i), \tag{10}$$

where Z_t is the aggregate productivity shock introduced in (4).

Since the intermediate goods substitute imperfectly for one another in producing the finished good, the representative intermediate goods-producing firm sells its output in a monopolistically competitive market, setting its nominal price $P_t(i)$ subject to the requirement that it satisfy the representative finished goods-producing firm's demand at that price. Following Rotemberg (1982), the intermediate goods-producing firm faces a quadratic cost of adjusting its nominal price between periods, measured in terms of the finished good and given by

$$\frac{\phi_p}{2}\left[\frac{P_t(i)}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}P_{t-1}(i)}-1\right]^2 Y_t,$$

where $\phi_p \geq 0$ governs the magnitude of the price adjustment cost, $\pi_{t-1} = P_{t-1}/P_{t-2}$ denotes the gross inflation rate during period $t-1$, and π denotes the steady-state rate of inflation. According to this specification, the extent to which price setting is backward-looking depends on the magnitude of the parameter α, which lies between zero and one, with $0 \leq \alpha \leq 1$. When, in particular, $\alpha = 1$, prices are fully indexed to past inflation, giving price setting an important backward-looking component. At the other extreme, when $\alpha = 0$, there is no indexation of prices to past inflation rates and price setting is purely forward-looking.

The cost of price adjustment makes the intermediate goods-producing firm's problem dynamic: it chooses $P_t(i)$ for all $t = 0, 1, 2, \ldots$ to maximize its total real market value. Section 1 of the appendix lists the first-order conditions for this problem.

2.5 The Efficient Level of Output and the Output Gap

A social planner for this economy who can overcome the frictions associated with monetary trade, sluggish price adjustment, and the monopolistically competitive structure of the intermediate goods-producing sector chooses Q_t and $n_t(i)$ for all $i \in [0,1]$ to maximize the social welfare function

$$E_0 \sum_{t=0}^{\infty} \beta^t a_t \left\{ \ln(Q_t - \gamma Q_{t-1}) - \frac{\chi_0}{1+\chi}\left[\int_0^1 n_t(i)\,\mathrm{d}i\right]^{1+\chi} \right\}, \tag{11}$$

subject to the aggregate feasibility constraint

$$Z_t\left[\int_0^1 n_t(i)^{(\theta_t-1)/\theta_t}\,\mathrm{d}i\right]^{\theta_t/(\theta_t-1)} \geq Q_t \tag{12}$$

for all $t = 0, 1, 2, \ldots$. The efficient level of output, defined as the value of Q_t that solves this problem according to the first-order conditions listed in section 1 of the appendix, implies a corresponding definition of the output gap as

$$x_t = Y_t/Q_t. \tag{13}$$

2.6 The Central Bank

The central bank in the estimated model conducts monetary policy according to a variant of the Taylor (1993) rule

$$\ln(r_t/r) = \rho_r \ln(r_{t-1}/r) + \rho_\pi \ln(\pi_t/\pi) + \rho_x \ln(x_t/x) + \varepsilon_{rt} \tag{14}$$

for all $t = 0, 1, 2, \ldots$, where, as above, π denotes the steady-state rate of inflation and, likewise, r and x denote the steady-state values of the gross nominal interest rate and output gap. This policy rule allows the central bank, through its choice of coefficients $\rho_\pi \geq 0$ and $\rho_x \geq 0$, to adjust the short-term nominal interest rate in response to movements in inflation and a model-consistent measure of the output gap.[3] It also allows for interest rate smoothing through the coefficient, satisfying $0 \leq \rho_r < 1$, on the lagged interest rate. The serially uncorrelated monetary policy shock ε_{rt} is normally distributed with mean zero and standard deviation σ_r.

Although the central bank itself makes no direct reference to the growth rate of money under an interest rate rule of this form, the model's money demand relationship serves to determine the equilibrium level of real balances M_t/P_t. The growth rate of nominal money

[3] Implicit in the formulation of this monetary policy rule, therefore, is the assumption that the central bank observes the output gap without a lag and with full accuracy. Once again, this assumption precludes any special role for the monetary aggregates in the New Keynesian model. It rules out, in particular, the usefulness of money in a "cross-check" that guards against output gap mismeasurement, a role considered in more detail by Beck and Wieland (2008).

$\mu_t = M_t/M_{t-1}$ is then determined by

$$\mu_t = \left(\frac{M_t/P_t}{M_{t-1}/P_{t-1}}\right)\pi_t \tag{15}$$

for all $t = 0, 1, 2, \ldots$. To keep track of the model's observable variables, it is useful to let

$$g_t = Y_t/Y_{t-1} \tag{16}$$

denote the growth rate of output for all $t = 0, 1, 2, \ldots$.

During the period from 2009 through 2015, when the Federal Reserve held the federal funds rate in a range close to zero, the Taylor rule (14) is replaced in the estimated model by the zero interest rate condition

$$\ln(r_t/r) = -\ln(r). \tag{17}$$

During this episode, changes in the stance of monetary policy are tracked, instead, by changes in the expected duration of the zero interest rate episode, reflecting the Fed's forward guidance, as described in more detail below.

To generate counterfactual outcomes under which monetary policy is described by a rule for the money growth rate, (14) is replaced instead by

$$\ln(\mu_t/\mu) = \rho_{mm}\ln(\mu_{t-1}/\mu) + \rho_{m\pi}\ln(\pi_t/\pi) + \rho_{mx}\ln(x_t/x), \tag{18}$$

where μ denotes the steady-state rate of money growth. When $\rho_{mm} = \rho_{m\pi} = \rho_{mx} = 0$, (18) reduces to the same constant money growth rule studied earlier by Ireland (2000), Collard and Dellas (2005), and Galí (2015) and advocated most famously, of course, by Friedman (1968). When, by contrast, $\rho_{m\pi} < 0$ and $\rho_{mx} < 0$, this more general money growth rate rule allows monetary policy to stabilize inflation and the output gap actively in response to other shocks that hit the economy. When, in addition, $\rho_{mm} > 0$, the rule prescribes a gradual response of money growth to movements in inflation and the output gap, in much the same way that the Taylor rule (14) with interest rate smoothing does.

2.7 Symmetric Equilibrium

Section 2 of the appendix collects the equations describing the model's symmetric equilibrium, in which all intermediate goods-producing firms make identical decisions, so that $Y_t(i) = Y_t$, $h_t(i) = h_t$, $D_t(i) = D_t$, and $P_t(i) = P_t$ for all $i \in [0, 1]$ and $t = 0, 1, 2, \ldots$. Some of the real variables in this system inherit unit roots from the random walk (4) in the technology shocks. However, the scaled variables $y_t = Y_t/Z_t$, $c_t = C_t/Z_t$, $m_t = (M_t/P_t)/Z_t$, $q_t = Q_t/Z_t$, and $z_t = Z_t/Z_{t-1}$ remain stationary and, in the absence of shocks, the economy converges to a steady-state growth path, along which all of the stationary variables are constant.

Section 2 of the appendix then log-linearizes the model's equilibrium conditions to describe how these stationary variables fluctuate around their steady-state values in response to shocks. Focusing on the special case where $\gamma = 0$ and $\alpha = 0$, so that there is no habit formation in consumption or indexation in price setting, these equations combine to yield the simpler, purely forward-looking New Keynesian IS and Phillips curves

$$\hat{x}_t = E_t\hat{x}_{t+1} - (r_t - E_t\hat{\pi}_{t+1}) + (1 - \rho_a)\hat{a}_t \tag{19}$$

and

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + (1 + \chi)\psi\hat{e}_t, \tag{20}$$

where, here and below, circumflexes denote variables expressed in logarithmic (percentage) deviations from their steady-state values. In the Phillips curve (20), the cost-push shock has been renormalized as $\hat{e}_t = -(1/\phi_p)\hat{\theta}_t$ and the composite parameter ψ defined as

$$\psi = (\theta - 1)/\phi_p. \tag{21}$$

Together with the Taylor rule (14), the IS and Phillips curves (19) and (20) determine the dynamics of the output gap, inflation, and the nterest rate without reference to the behavior of the money stock, real or nominal. Adding habit formation and price indexation enriches those dynamics, but does not overturn this basic property of the New Keynesian model: the effects that monetary policy has on output and inflation are summarized completely by the current and expected future path of the short-term nominal interest rate.

The addition of the model's money demand relationship, however, serves to determine the equilibrium level of real balances under the Taylor rule (14). Focusing again on the special case where $\gamma = 0$ and $\phi_m = 0$, so that there is no habit formation in consumption or adjustment costs for real balances, the log-linearized equations from section 2 of the appendix also imply that

$$\hat{m}_t = \delta_r(r-1)\hat{y}_t - \delta_r\hat{r}_t + \hat{u}_t, \tag{22}$$

where the composite parameter δ_r is determined by

$$\delta_r = \chi_0 \left(\frac{\delta}{r}\right)\left(\frac{\theta}{\theta-1}\right) y^{\chi}, \tag{23}$$

and y is the steady-state value of $y_t = Y_t/Z_t$. The coefficient on $\hat{y}_t$ in (22) will be small for modest levels of the steady-state nominal interest rate; hence, in this economy, real money demand depends on the permanent component of income, captured by Z_t, more than the transitory component $\hat{y}_t$. Interpreting (22) as a long-run money demand curve in the more general case where habit formation and adjustment costs for real balances reappear, δ_r measures the interest semi-elasticity of long-run money demand.

3 Solution and Estimation Methods

3.1 Accounting for Forward Guidance

Kulish, Morley, and Robinson (2017) develop the methods used to solve and estimate the New Keynesian model described above while accounting for the additional monetary stimulus delivered through the Federal Reserve's forward guidance while the federal funds rate itself was held in a range near zero from 2009 through 2015. These methods are outlined in detail in parts 3-6 of the appendix; they are summarized here.

Part 3 of the appendix begins by rewriting the model's log-linearized equilibrium conditions, away from the lower bound when monetary policy is conducted according to the Taylor rule (14), as

$$A_0 s_{0,t} = A_1 s_{0,t-1} + B_0 E_t s_{0,t+1} + C_0 \xi_t \tag{24}$$

and

$$\xi_t = P\xi_{t-1} + \varepsilon_t, \tag{25}$$

where the vectors $s_{0,t}$, ξ_t, and ε_t keep track of the model's endogenous variables, exogenous shocks, and associated innovations, and the matrices A_0, A_1, B_0, C_0, and P have elements that depend on the model's structural parameters. Part 3 of the appendix shows that this system has solution

$$s_{0,t} = Ds_{0,t-1} + H\xi_t, \tag{26}$$

where the elements of the matrices D and H depend, nonlinearly, on the elements of the matrices in (24) and (25). Equation (26) is used to describe the model's dynamics during the periods before and after the zero interest rate episode.

Part 4 of the appendix replaces the Taylor rule (14) with the zero nominal interest rate condition (17), so that the system of linearized equations collected previously in (24) takes the alternative form

$$\bar{A}_0 s_{0,t} = \bar{J}_0 + \bar{A}_1 s_{0,t-1} + \bar{B}_0 E_t s_{0,t+1} + \bar{C}_0 \xi_t. \tag{27}$$

Let $t = T_1$ denote the start of the zero interest rate period and suppose that, upon replacing (14) with (17), the central bank also provides forward guidance through an announced commitment to hold interest rates at zero for the next τ periods, even if the Taylor rule prescribes a positive interest rate. Part 4 of the appendix then shows that during periods $t = T_1, T_1+1, \ldots, T_2 = T_1+\tau-1$, the endogenous variables in $s_{0,t}$ will follow a solution with time-varying coefficients:

$$s_{0,t} = J_t + D_t s_{0,t-1} + H_t \xi_t, \tag{28}$$

where, starting from the terminal conditions $D_{T_2+1} = D$, $H_{T_2+1} = H$, and $J_{T_2+1} = 0$, the sequences $\{D_{T_1+j}\}_{j=0}^{\tau-1}$, $\{H_{T_1+j}\}_{j=0}^{\tau-1}$, and $\{J_{T_1+j}\}_{j=0}^{\tau-1}$ can be found via backward recursion.

Now assume, more generally, that the central bank re-evaluates the timing of its return to conventional policymaking via the Taylor rule (14) each period throughout the zero interest rate episode, announcing at the beginning of t that rates will be held at zero for τ_t more periods. To characterize outcomes in this case, let $\bar{\tau}$ be an arbitrarily large upper bound on the length of the zero interest rate episode, and re-label the subscripts on the time-varying matrices so that $\{D_k\}_{k=1}^{\bar{\tau}}$, $\{H_k\}_{k=1}^{\bar{\tau}}$, and $\{J_k\}_{k=1}^{\bar{\tau}}$ are those that apply during any period when the zero interest rate episode is expected to last for k more periods. Now, the matrices that appear in the solution (28) are given by $D_t = D_{\tau_t}$, $H_t = H_{\tau_t}$, and $J_t = J_{\tau_t}$ for each period t during the zero interest rate episode.

Evaluating the likelihood function and simulating the posterior distribution for the model, based on the solution described by (26) and (28), is complicated by three factors. First, because the short-term nominal interest rate has zero variance according to the zero interest rate condition (17), it must be removed from the list of observables during the zero interest rate episode. Second, the matrices entering into the state-space model implied by (25), (26), and (28) are time-varying. These two complications can both be accommodated within the standard Kalman filtering framework, as shown by Harvey (1989, Ch.3) and Anderson and Moore (2005, Ch.3) and outlined in section 5 of the appendix.

Third, in addition to the structural parameters that enter into the New Keynesian model's equilibrium conditions, which can be collected into a vector Θ, there is now a second set of parameters

$$\Delta = \{\tau_{T_1}, \tau_{T_1+1}, \dots, \tau_{T_2}\}$$

for the expected duration of the zero interest rate episode at each period during that episode, starting at date T_1 and ending at T_2. The model's log posterior kernel must therefore be evaluated as

$$\ln L(\{d_t\}_{t=1}^{T} | \Theta, \Delta) + \ln(P(\Theta, \Delta)),$$

where L denotes the likelihood function based on the sample of data $\{d_t\}_{t=1}^{T}$ and $P(\Theta, \Delta)$ is the prior density over both sets of parameters. Kulish, Morley, and Robinson's (2017) modification of the randomized block Metropolis-Hastings algorithm of Chib and Ramamurthy (2010), described in section 6 of the appendix, is used to simulate draws from this posterior distribution. The algorithm treats Θ and Δ as separate blocks of parameters, as those in Θ are continuously-valued whereas the durations in Δ are restricted to the positive integers.

Kulish, Morely, and Robinson's (2017) solution procedure resembles Guerrieri and Iacoviello's (2015) in that both rely on piecewise linear approximations to the model's equilibrium condition under a "reference regime" (24) in which the nominal interest rate is positive and an "alternative regime" (27) in which the nominal interest rate equals zero. In Guerrieri and Iacoviello's (2015) framework, however, switching between these two regimes is governed by whether the Taylor rule (14) calls for a positive interest rate: if so, the economy remains in the reference regime and, if not, the economy switches to the alternative regime. Under Kulish, Morley, and Robinson's (2017) algorithm, by contrast, regime switches occur depending on whether, in the dataset used to estimate the model, the nominal interest rate remains positive or equals zero. In the latter case, the expected duration of the zero interest rate episode gets determined through the central bank's forward guidance and may extend beyond the period over which the Taylor rule (14) calls for rates at or below zero. Through this key difference, the approach accounts explicitly for the extra monetary stimulus applied when, through forward guidance, the central bank promises to hold rates "lower for longer."

Compared to the fully nonlinear methods of Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2015), Gust, Herbst, López-Salido, and Smith (2017), and Aruoba, Cuba-Borda, and Schorfheide (2018), Kulish, Morley, and Robinson's (2017) algorithm ignores two types of anticipation effects. First, although time variation in the parameters of (28) reflect agents' expectations of eventual lift-off from the central bank's perceived zero lower bound, once the economy has returned to the reference regime with positive interest rates, consumers and firms do not account for the possibility of a future episode of zero interest rates. Second, although the algorithm is designed specifically to account for the effects of forward guidance, consumers and firms repeatedly adjust their expectations for the duration of the zero interest rate episode, treating each change as a once-and-for-all shift, without accounting for the possibility that the announced exit date may shift again in the future.[4] In accepting these limitations, Kulish, Morley, and Robinson's (2017) algorithm remains tractable and efficient enough to facilitate Bayesian estimation of the model's structural parameters and allows survey data on interest rate expectations to help gauge the effect that the Federal Reserve's policies of forward guidance had on output and inflation during the 2009-15 episode of zero interest rates.

3.2 Data and Priors

All data used to estimate the model are quarterly and run from 1983:1 through 2019:4. For the periods from 1983:1 through 2008:4 and from 2016:1 through 2019:4 during which the Federal Reserve was targeting the federal funds rate at levels bounded away from zero, the

[4] These same limitations apply in Del Negro, Giannoni, and Schorfheide's (2015) analysis, where changing expectations driven by the Fed's forward guidance is modeled in a similar way.

estimation procedure treats four of the model's variables as observable. Output growth is measured by quarterly changes in the natural log of real GDP, converted to per capita terms using the civilian noninstitutional population, age 16 and over.[5] Inflation is measured by quarterly changes in the log of the GDP deflator, and the short-term nominal interest rate is measured by the effective federal funds rate, divided by 400 to convert the annualized figures in the data to quarterly rates as they appear in the model. Finally, nominal money growth is measured by quarterly changes in the Divisia M2 index of monetary services, again converted to per capital terms. Serletis and Gogas (2014) and Belongia and Ireland (2019) find evidence of stable long-run money demand relationships for Divisia M2, motivating the choice of that monetary aggregate here.[6] All data series are drawn from the Federal Reserve Bank of St. Louis' FRED database, except that for Divisia M2, which is available through the Center for Financial Stability's website. For details on the construction of the CFS Divisia monetary series, see Barnett, Liu, Mattson, and van den Noort (2013).[7]

For the period from 2009:1 through 2015:4 during which the Federal Reserve held short term interest rates close to zero, (17) replaces (14) in the estimated model, and the federal funds rate is dropped from the list of of observable variables. For this interval, the model's solution depends not only on the structural parameters that enter into the New Keynesian model, but also on the duration of the zero interest rate episode expected by private agents

[5]In light of the erratic behavior, noted by Edge and Gurkaynak (2010, p.218) and caused by periodic rebasings of the Census Bureau series for the US population, the trend values for this series, extracted using the Hodrick-Prescott filter with smoothing parameter 1600, are used in place of the raw data in constructing per-capita measures here.

[6]More generally, Belongia (1996), Barnett and Chauvet (2011), Hendrickson (2014), Belongia and Ireland (2015*a*, 2016), and Ellington (2018) all find evidence of tighter statistical links, more consistent with the implications of macroeconomic theory, between key macroeconomic variables and Divisia monetary aggregates compared to their simple-sum counterparts. For a detailed overview of the theory of Divisia monetary aggregation, see Barnett (2012).

[7]During the third quarter of 2011, both Divisia M2 and the Federal Reserve's official, simple-sum measure of M2 grew by more than 15 percent on an annualized basis, reflecting two regulatory changes discussed by Judson, Schlusche, and Wong (2014, pp.11-2). First, changes in FDIC insurance rates helped banks bring offshore deposits back to the US and, second, the lifting of the longstanding prohibition against paying interest on demand deposits triggered additional portfolio shifts back into M2. To prevent this large spike in money growth, which could have been anticipated in advance, from distorting estimates of the model's parameters, the observed rate of money growth for 2011:3 is replaced by the average of the rates of money growth for 2011:2 and 2011:4.

in the model during each period t. Following Kulish, Morley, and Robinson (2017) and as described above, these expected durations are treated as parameters that can be estimated based on the forward-looking New Keynesian model's interpretations for the effects that expected future interest rates have on the remaining observables: output growth, inflation, and the nominal money growth rate.

Prior to estimation, values for z, π, and β are fixed at values that match the model's steady-state output growth, inflation, and short-term nominal interest rate to the average values of those same variables in the pre-crisis subsample of data running from 1983:1 through 2008:4. Likewise, the steady-state money growth rate μ is treated as another free parameter, fixed to match the average growth rate of Divisia M2 over the same 1983:1-2008:4 period. Thus, in the estimation, all four observable series are re-expressed as deviations from their pre-crisis mean values. This approach is intended, in particular, to force the estimated model to attribute the Great Recession of 2007-9 and the subsequent sluggish recovery to one or more highly persistent shocks instead of downward shifts in the steady-state rates of output growth and inflation. Bayesian priors must then be specified and calibrated for two vectors of parameters: the remaining structural parameters from the New Keynesian model and the time-varying expected durations of the zero nominal interest rate episode over the 28 quarters from 2009:1 through 2015:4.

Priors for the 17 structural parameters are summarized in table 1 and described in more detail by the red lines in figures 1 and 2. The habit formation and price indexation parameters γ and α, the interest rate smoothing parameter ρ_r, and the autoregressive parameters ρ_a, ρ_u, and ρ_e all lie between zero and one. Independent beta prior distributions are therefore assigned to these parameters, each with its two shape coefficients calibrated to match the prior mean and standard deviation listed in table 1. In particular, prior distributions for γ and α are centered at 0.5, with standard deviations large enough to allow for values closer to zero or one. The prior distributions for ρ_r, ρ_a, ρ_u, and ρ_e reflect more confidence in the beliefs, first, that Federal Reserve policy over the sample period is characterized by substantial interest rate smoothing and, second, that while the preference and money demand shocks hitting the economy are highly persistent, the cost push shocks are less so.

The inverse labor supply elasticity χ, the Phillips curve slope ψ defined in (21), the interest semi-elasticity of money demand δ_r from (23), a rescaled adjustment cost parameter

$$\phi = \delta\phi_m \tag{29}$$

for real balances defined in part 2 of the appendix when log-linearizing the model, and the monetary policy coefficients ρ_π and ρ_x are all nonnegative. Independent gamma prior distributions are assigned to these parameters, with shape and scale coefficients chosen to match the means and standard deviations listed, again, in table 1. The prior mean of 1.4 for χ is based on Aruoba, Cuba-Borda, and Schofheide's (2018) calibrated value of 1/0.72; the relatively large standard deviation acknowledges the uncertainty, emphasized by Ríos-Rull, Schorfheide, Fuentes-Albero, Kryshko, and Santaeulàlia-Llopis (2012), surrounding previous estimates of the Frisch labor supply elasticity. The prior mean for ψ is set equal to 0.1, the calibrated value used by Ireland (2000, 2004*b*, 2004*c*, 2007, 2011). Belongia and Ireland (2019) estimate cointegrating money demand relationships for Divisia M2 with semi-elasticities of approximately 3.75. Here, with interest rates measured in quarterly instead of annual terms, a prior mean of 15 for δ_r is fixed by multiplying that value by 4. A prior mean of 10 ϕ is chosen somewhat arbitrarily, given the lack of out-of-sample evidence on this size of this money demand adjustment cost parameter. Large standard deviations for δ_r and ϕ then reflect, once again, the considerable uncertainty surrounding the magnitudes of these parameters before estimation. Prior means of 0.4 for ρ_π and 0.2 for ρ_x imply a policy response to changes in inflation twice the size of the response to changes in the output gap. Coupled with the prior mean of 0.75 for the interest rate smoothing parameter ρ_r, these settings translate into long-run responses $\rho_\pi/(1-\rho_r)$ and $\rho_x/(1-\rho_r)$ of the interest rate to changes in inflation and the output gap equal to 1.6 and 0.8.

Finally, the parameters σ_a, σ_z, σ_u, σ_e, and σ_r measuring the standard deviations of the New Keynesian model's five structural shocks are assigned prior distributions implied by the assumption that each of the associated variances has an inverse chi-squared distribution with four degrees of freedom.[8] The scale coefficient for the inverse chi-squared distribution is set to 0.01^2 for the preference, productivity, and money demand shocks and 0.0025^2 for the cost push and monetary policy shocks. These settings give σ_a, σ_z, and σ_u prior means equal to 0.0125 and σ_e and σ_r prior means of 0.0031. The implied standard deviations, meanwhile, leave considerable leeway for the data to push the posterior distributions towards larger or smaller values of each of these volatility parameters.

Priors for the time-varying expected duration of the zero nominal interest rate episode are formed, as described by Kulish, Morley, and Robinson (2017, p.40), with the help of data from the Blue Chip Financial Forecasts from 2009:1 through 2010:4 and the Federal Reserve Bank of New York Primary Dealers Survey from 2011:1 through 2015:4. The Blue Chip survey records the expected date of funds rate liftoff from zero reported by approximately 40 forecasting firms; the cross-sectional distribution of these forecasts is interpreted as reflecting the probabilities of various durations of the zero interest rate episode. The Primary Dealers Survey, meanwhile, asks each individual respondent to assign probabilities to different liftoff dates; the average probabilities, as reported by the New York Fed, are used, similarly, as measures of the probabilities of different durations. Kulish, Morley, and Robinson (2017) form their prior as an equally-weighted average of the probabilities implied by these surveys and a uniform distribution over all durations ranging from 1 to 23 quarters.[9] Their estimation exercise, however, uses data on the term structure of interest rates as well as macroeconomic variables to glean additional information about expected durations of the zero interest rate episode using Smets and Wouters' (2007) medium-scale New Keynesian model, augmented with a model of the yield curve based on the expectations hypothesis. Here, where only

[8] Adjemian (2016) refers to this induced distribution as the "inverted gamma distribution of type I."

[9] The upper bound of 23 quarters imposed by the prior on the expected duration of the zero interest rate episode reflects the observation that, in the survey data, zero probabilities are assigned to all durations longer than 23 quarters.

macroeconomic data are used for estimation, a larger weight of 80 percent is assigned to the survey evidence, with the remaining 20 percent attached to a uniform distribution over durations from 1 to 23 quarters. The resulting independent prior distributions for all of the 28 expected durations prevailing from 2009:1 through 2015:4 are plotted as red lines in figures 3 and 4.

4 Bayesian Estimates

Table 2 summarizes the posterior distributions of the New Keynesian model's 17 structural parameters, while figures 1 and 2 display more fully the posterior densities using blue bars, comparing them to the priors, described above and outlined in red. These posterior distributions assign more weight to higher values of the habit formation parameter γ and lower values for the price indexation parameter α, compared to the priors. The posterior densities for χ and ψ imply a more elastic labor supply and a much flatter Phillips curve than do the priors, perhaps reflecting the muted response of inflation to the more dramatic movements in real variables during and since the Great Recession.[10]

Posteriors for the parameters ρ_r, ρ_π, and ρ_x from the Taylor rule (14) imply an even larger degree of interest rate smoothing and a more balanced response of policy to changes in inflation and the output gap than suggested by the prior. Table 3 reports additional results to help assess the robustness of these findings. In each case, the entire model is re-estimated; to economize on space, however, only the estimates of the parameters of the monetary policy rule are shown. Table 3 shows, first, that when the contemporaneous money growth rate is added to the list of variables on the right-hand side of (14), the model's log marginal likelihood increases, but only slightly, and the estimated response coefficient ρ_μ

[10] The formula displayed by Del Negro, Giannoni, and Schorfheide (2015, p.174) can be used together with information displayed in table A-2 of the appendix to that same paper to compute the Phillips curve slope coefficient (labeled κ) implied by the posterior mode from estimating both Smets and Wouters' (2007) medium-scale New Keynesian DSGE model and an extended version featuring additional financial frictions developed specifically to explain the behavior of inflation over the post-crisis period. The posterior mode at $\psi = 0.0153$ found here is comparable to the modal value of $\kappa = 0.0120$ from the Smets-Wouters model but substantially larger than the modal value of $\kappa = 0.0018$ from the extended model with financial frictions.

on money growth is small.[11] The table shows, second, that the marginal likelihood falls substantially when lagged values of inflation, the output gap, and money growth replace the contemporaneous values that appear, instead, in the benchmark (14). Taken together, these findings point to the adequacy of (14) in describing the Fed's interest rate policy away from the zero bound.

In table 2 and figure 2, estimates of ρ_a and σ_a suggest that non-monetary aggregate demand disturbances have been large and persistent over the sample period. Estimates of ρ_u and σ_u, meanwhile, show that even more persistent money demand shocks have been important, too. Earlier results from Ireland (2000), Collard and Dellas (2005), and Galí (2015) suggest that these money demand shocks will become an important source of additional macroeconomic volatility when the estimated Taylor rule is replaced by one calling for a constant rate of money growth. Less certain, however, is whether a money growth rule of the more general form (18) can cope more successfully with these disturbances. Finally, the posterior density for σ_z, measuring the volatility of productivity shocks, tightens but remains centered near its prior mean, while the volatility parameters σ_e and σ_r for the cost-push and monetary policy shocks appear smaller, relative to values initially suggested by the prior.

Figures 3 and 4 show that the posterior distributions for the expected durations of the zero nominal interest rate episode overlap heavily with the corresponding priors, reflecting the absence of the additional term structure data that Kulish, Morley, and Robinson (2017) use to sharpen their estimates of these parameters. While the macroeconomic data do contribute modestly to determining the shape of these posterior distributions, to a large extent the expected durations here are essentially calibrated based on the survey data used to formulate the priors. Even by themselves, however, these survey data are useful in incorporating into the estimated model the shift in expectations towards much longer durations of the zero

[11] For the rules including money growth, the prior distribution for the response parameter ρ_μ is gamma with mean 0.2 and standard deviation 0.1; hence, as shown in the table, the data push the posterior distribution close to zero regardless of whether contemporaneous or lagged variables enter the Taylor rule. In all cases, the marginal likelihood is computed using Geweke's (1999) modified harmonic mean, modified as described in section 7 of the appendix to account for the distinction between the elements of Θ, which are continuous-valued and those of Δ, with values restricted to the positive integers.

nominal interest rate episode that Swanson and Williams (2014) observe in late 2010, as well as the gradual reduction in expected durations as the economy continued to recover in 2014 and 2015.

With five structural disturbances, the New Keynesian model estimated here can track all four observables without the need for measurement or specification errors of the kind considered, for instance, by Ireland (2004*a*). Thus, it is not meaningful to ask how well the model in this sense "fits" the data: it tracks the data perfectly, period by period. It is instructive, however, to look in various ways at how the estimated model breaks the historical volatility in each observable down into components attributable to the various shocks.

Table 4 does so by decomposing the variance in selected variables, using the solution (26) that applies away from the zero interest rate bound, where the monetary policy shock in the Taylor rule (14) is operative and forward guidance is not relevant. The table reports the means instead of the medians of the posterior distributions for the fractions of each variable attributable to each shock so that the fractions across rows sum to one. The table reveals that fluctuations in output growth get driven mainly by technology shocks, in inflation mainly by cost-push shocks, and in the nominal interest rate mainly by preference shocks. About 60 percent of the volatility in money growth is attributable to the Fed's deliberate monetary policy response, under the Taylor rule (14), to fluctuations in the output gap and inflation, which are driven more fundamentally by shocks to preferences and technology. More than 35 percent of money growth variation, however, occurs under the Taylor rule (14) to offset shocks to money demand. This result suggests that under a constant money growth rate rule, these money demand shocks will translate, instead, into unwanted volatility in output and inflation – a result that will be confirmed below.

Figure 5 turns this exercise around, and plots the median paths from the posterior distributions of the structural disturbances themselves.[12]. Not surprisingly, the estimated model attributes the Great Recession, with its accompanying declines in inflation and interest rates,

[12]These paths are based on draws from the posterior distribution for each shock, taken using Durbin and Koopman's (2002) simulation-smoother for the unobservable states, as described in part 8 of the appendix.

to a series of large, adverse preference shocks. Unfavorable productivity shocks also appear throughout the post-2008 period, contributing to weakness in real GDP growth but also explaining why inflation did not fall even further. This characterization of the Great Recession as the product of adverse preference and technology shocks is also implied by estimates of Aruoba, Cuba-Borda, and Schorfheide's (2018) small-scale New Keynesian model. It is broadly consistent, too, with the implications of larger-scale models estimated by Campbell, Fisher, Justiniano, and Melosi (2016) and Gust, Herbst, López-Salido, and Smith (2017). Those larger-scale models distinguish explicitly between consumption and investment, however, where a preference shock like the one that appears here moves these two key components of real GDP in opposite directions. Instead, a "risk-premium" shock, introduced in these models as a shock to the weight on holdings of risk-free bonds in the representative household's utility function, enters into the household's Euler equation but also generates co-movement in consumption and investment. In both of those larger-scale models, this alternative shock to preferences combines with adverse technology shocks to account simultaneously for the sharp declines in output and interest rates and the modest fall in inflation observed during and immediately after the Great Recession. And while they prefer an alternative account in which inflation dynamics are explained, instead, by expectations of future fiscal policy actions, Bianchi and Melosi (2017) also show that a standard New Keynesian model can explain key features of the Great Recession by appeal to a combination of adverse preference and technology shocks.

The middle row of figure 5 plots the median paths for both the money demand shock $\hat{u}_t$ in levels and the serially uncorrelated innovation ε_{ut} to this highly persistent shock. The sharp decline in $\hat{u}_t$ during the early 1990s coincides with the period of "missing M2" analyzed by Duca (2000), which as noted by Orphanides and Porter (2000) worked to throw off track the predictions of Hallman, Porter, and Small's (1991) "P-star" model, given its assumption that M2 velocity would converge more quickly back to a constant long-run mean.[13] The estimated

[13]The first footnote in Hallman, Porter, and Small (1991) credits Federal Reserve Chair Alan Greenspan for suggesting the idea behind the P-star model. Thus, that model is of historical interest as one of the last

path for $\hat{u}_t$ remains stable over a period extending from the mid-1990s through 2008, before moving sharply higher during and after the Great Recession of 2007-9. Anderson, Bordo, and Duca (2017) detect a similar increase in post-2008 M2 demand, relative to what would be expected based on movements in income and interest rates, which they attribute to flight-to-quality portfolio dynamics triggered by the financial crisis. With specific focus on repairing the P-star model, however, Belongia and Ireland (2015*b*, 2017) show that these shifting but highly persistent trends in money demand need not present an obstacle to effective policymaking via money growth targeting. Precisely because the trends are so persistent, they can be recognized by policymakers in real time, and successfully accommodated by gradual shifts in the target for money growth so as to stabilize output growth, inflation, and nominal GDP growth as their sum. More problematic are the innovations ε_{ut} to the money demand shock, which are unpredictable from past data and therefore more likely to contribute to volatility, especially under constant money growth, but perhaps also under more general rules of the form in (18).

Finally, figure 6 illustrates the effects that extended forward guidance has on inflation and the output gap according to the estimated model. As Koop, Pesaran, and Potter (1996) explain, impulse responses are no longer independent of the state of the economy, the history of past shocks, or the size and sign of the particular shock considered, in nonlinear models. Thus, the details of how impulse responses are constructed from the nonlinear solution based on (28) matter. Here, in particular, each panel shows the percentage-point difference between the median path for the indicated variable when the duration of the expected zero interest rate episode is extended by one additional quarter at the indicated date from the median path for the same variable when the duration is held at its estimated value.

The estimated effects of forward guidance are smallest at the beginning and end of the actual zero interest rate episode when, as shown previously in figures 3 and 4, the estimated

quantity-theoretic models consulted regularly by high-level officials at the Federal Reserve Board. Similarly, the paper itself remains noteworthy as one of the last quantity-theoretic studies to be published in a leading, general-interest economics journal.

durations are short. They are largest from 2011 through 2013 when, by contrast, the estimated durations extend out two or three years. For the fourth quarter of 2013, in particular, the graphs indicate that a further one-quarter increase in the expected duration of the zero interest rate episode would have lifted inflation by almost 50 basis points and the output gap by more than 25 basis points. Although these estimates confirm Carlstrom, Fuerst, and Paustian (2015) and Del Negro, Giannoni, and Patterson's (2015) findings that, in New Keynesian models, the effects of forward guidance grow stronger as the expected duration of the zero interest episode lengthens, they also confirm Harrison (2015) and Campbell, Fisher, Justiniano, and Melosi's (2016) findings that these effects do not become unreasonably large when the durations considered remain close to what, in practice, the Federal Reserve promised through its actual forward guidance over the 2009-15 episode of zero interest rates.

5 Policy Analysis

5.1 A Flexible Money Growth Rule

Table 5 begins to compare the US economy's actual performance under the estimated Taylor rule to its hypothetical performance under counterfactual money growth rules. In the estimated model, output growth, inflation, the nominal interest rate, and the money growth rate are all observable and, as noted above, the model can track these variables perfectly over the sample period. Hence, the Bayesian estimation procedure treats the historical standard deviations of those same variables as observable as well; therefore, these are constant across the first set of columns of the table's top panel. The New Keynesian model's output gap $\hat{x}_t$, however, remains unobservable to us, as econometricians; therefore, its historical behavior must be estimated together with the model's other parameters.[14]

Although the earlier studies by Ireland (2000), Collard and Dellas (2005), and Galí

[14] Again, paths for the output gap are constructed from draws taken using Durbin and Koopman's (2002) simulation-smoother for the model's unobservable states.

(2015) all found that a constant money growth rule produced excess volatility after money demand shocks relative to an interest rate rule, none of them considered the alternative of a money growth rule that adapts flexibly to changing macroeconomic conditions in the same manner as the Taylor rule. Relative to the Taylor rule, in fact, one potential advantage to more flexible money growth rules of the form shown in (18) is that they do not require the aggressive response to inflation needed by interest rate rules to ensure the stability of a unique rational expectations equilibrium. Instead, money growth rules can stabilize long-run inflation simply by pinning down the average rate of money growth and focus directly on stabilizing the output gap over shorter time horizons.

Though no exhaustive attempt has been made here to identify the optimal money growth rule, search over a grid of values for the parameters reveals that setting $\rho_{mm} = 1$, $\rho_{m\pi} = 0$, and $\rho_{mx} = -0.125$ delivers impressive performance over the 1983:1-2019:4 sample period, while minimizing the duration and importance of the episode, during and following the financial crisis and Great Recession, over which the short-term nominal interest rate fluctuates in a range near zero. This rule, which specializes (18) as

$$\hat{\mu}_t = \hat{\mu}_{t-1} - 0.125\hat{x}_{t-1}, \tag{30}$$

generates modest but highly persistent adjustments in money growth. These adjustments work, directly, to stabilize the output gap and, indirectly, to stabilize inflation as well.

The second set of columns in the top panel of table 5 summarize the posterior distributions of output growth, inflation, the nominal interest rate, the money growth rate, and the output gap under the flexible money growth rule (30), holding all other parameters and disturbances fixed at their estimated values. Thus, these counterfactual simulations confront the central bank with the same patterns of preference, productivity, money demand, and cost push shocks estimated to have hit the US economy over the 1983:1-2019:4 sample period, but replace the Federal Reserve's historical policy of interest rate management, including forward guidance used to lengthen the expected duration of the zero nominal interest rate episode, with the policy dictated by the flexible money growth rule instead.

As noted above, the form of the model's money demand relationship allows the nominal interest rate to fall below zero in a well-defined rational expectations equilibrium. Within the model, the preference specification (7) imposes welfare costs of negative interest rates on the household; Rognlie (2016) studies these costs in greater detail. Outside the model, concerns might arise that if the interest rate were to fall below zero for an extended period of time, the private financial system would adapt to profit from the spread between the zero interest rate on currency and the negative nominal interest rate on bonds, perhaps in ways that impose additional costs on the economy. It will be confirmed below, however, that under (30), the episode of negative nominal interest rates is moderate and, in fact, considerably shorter than the seven-year period during which the Federal Reserve kept its federal funds rate target in a range close to zero.

Table 5 reveals that the flexible money growth rule (30) holds the volatility of output growth and inflation at levels closely approximating those achieved under the estimated rule. The output gap becomes more volatile under (30), but not dramatically so. Figures 7-10 add detail, by illustrating how that rule allows the output gap and inflation to respond to shocks in ways that resemble how they behave under the estimated interest rate rule. These figures compare the impulse responses under the estimated interest rate rule and the flexible money growth rule of output growth, inflation, the nominal interest rate, the money growth rate, and the output gap to one-standard-deviation preference, productivity, money demand, and cost push shocks.[15]

Figure 7, in particular, reveals that both the estimated interest rate rule and the flexible money growth rule produce a monetary tightening that virtually eliminates the inflationary effects of an expansionary preference shock; the same monetary tightening helps stabilize the output gap as well. In figure 8, meanwhile, the estimated interest rate rule stabilizes

[15]For the estimated model, these impulse responses are based on the solution (26) that applies away from the perceived zero lower interest rate bound.

inflation following a productivity shock; to do so, it produces the increase in money growth that Ireland (1996) shows is necessary to generate, under sticky prices, the efficient increase in output that keeps the output gap unchanged. Likewise, the flexible money growth rule (30) calls for a monetary expansion after a favorable productivity shock that minimizes its impact on inflation and the output gap.

Figure 9 confirms that here, as in Poole's (1970) classic Keynesian analysis, the estimated interest rate rule, by holding the short-term nominal interest rate fixed, insulates output growth, inflation, and the output gap by fully accommodating a shock to money demand. The flexible money growth rule falls short of achieving this goal, but nevertheless generates a persistent increase in money growth that largely accommodates the increase in money demand. Finally, figure 10 shows impulse response to cost-push shocks under (30) that come close to replicating those that appear under the estimated interest rate rule.[16]

5.2 Constant Money Growth

Consistent with the earlier results from Ireland (2000), Collard and Dellas (2005), and Galí (2015), the results in the first set of columns in the bottom panel of table 5 suggest strongly that macroeconomic volatility would have been amplified greatly if the Federal Reserve had followed a policy directed at holding the growth rate of Divisia M2 perfectly fixed by setting $\rho_{mm} = \rho_{m\pi} = \rho_{mx} = 0$ in (18), again holding all other parameters and disturbances fixed at their estimated values. The median estimate of the standard deviation of output growth under the constant money growth rule is more than 50 percent larger than that under the

[16]A comparison between the first columns of figures 8 and 10 also sheds light on how the estimated model distinguishes between its two supply-side disturbances – the productivity and cost-push shocks – both of which move output and inflation in opposite directions on impact. This distinction is made, first, because the random walk specification in (4) allows the productivity shock to have permanent effects on the level of output, whereas the stationary specification in (9) implies that cost-push shock's effects are only transitory: this can be seen by observing that the initial decline in output growth in figure 10 is followed by a period of above-average growth. The distinction is made, second, because the productivity shock affects the model's efficient level of output while the cost-push shock does not; then, since the estimated policy rule (14) includes a response to the output gap, the interest rate and money growth rate respond differently to the two disturbances. Thus, more generally, the DSGE model successfully identifies all five structural disturbances using information in the four observable variables.

estimated policy rule. Meanwhile, volatility in inflation nearly doubles, and the standard deviation of the output gap increases by nearly a factor of four.

Figures 7-10 again add detail. In figure 7, the monetary tightening prescribed by both the estimated interest rate and the flexible money growth rule does not occur under the constant money growth rule. Hence, under constant money growth, output growth, inflation, and the output gap all display considerably more volatility in response to preference shocks. Similarly, in figure 8, the increase in money growth that helps the economy respond more efficiently to a productivity shock under both the estimated interest rate rule and the flexible money growth rule is not generated by the constant money growth rule. Hence, conditional on the productivity shock, output growth is perversely more stable under the constant money growth rule, even as inflation and the output gap become more volatile. As expected, figure 9 shows that the constant money growth rule, quite unlike the flexible money growth rule, allows money demand shocks to contribute greatly to macroeconomic volatility. Finally, in figure 10, the constant money growth rule does a slightly better job than the estimated interest rate rule and the flexible money growth rule of stabilizing inflation in response to an unfavorable cost-push shock, at the cost of allowing for much greater volatility in output growth and the output gap. Overall, the performance of the constant money growth rule appears quite poor, relative to both the estimated interest rate rule and the more flexible money growth rule (30).

5.3 Counterfactual Simulations

Figure 11 helps complete the analysis by plotting the median paths for output growth, inflation, the nominal interest rate, the money growth rate, and the output gap from the posterior distributions implied by the estimated model and the two counterfactual money growth rules. The panels in its third column confirm that measured by output growth or the output gap, the Great Recession would have been considerably more severe under the constant money growth rule. Inflation would not only have been more volatile, but the US economy would have experienced recurring, though brief, episodes of deflation even before the financial crisis in 2008. Under constant money growth, however, the short-term nominal interest rate would have fallen below zero for only three quarters between 2009:2 and 2010:1, reaching a low of −0.65, still higher than the target maintained by the Swiss National Bank over the entire period since 2015, in 2009:3.

The flexible money growth rule (30), again by sharp contrast, would have closed the negative output gap by the end of 2009. The flexible rule produces a smoother time path for money growth than that observed historically. Most important, it requires only two quarters of negative interest rates. Along the counterfactual path, the short-term interest rate drops to −0.97 percent in 2010:1, before rebounding to 0.27 percent in 2010:2 then falling back to −0.01 in 2010:3. From 2010:4 on, interest rates return to levels prevailing through much of the 1990s and 2000s. Additionally, under (30), inflation remains positive, both during and after the financial crisis. The two panels in the left-hand column on figure 12 provide additional detail in the comparison between the estimated policy rule and the flexible money growth rule, by bracketing the median paths from the posterior distribution of each variable with the corresponding values at the 16th and 84 percentiles.[17] Both graphs confirm that the flexible money growth rule shifts the entire distribution of outcomes for the output gap and inflation noticeably higher over the period since 2008.

One perennial criticism of a monetary policy strategy based on money growth is that it would produce excessive volatility in interest rates, In figure 11, the nominal interest rate does exhibit a modest degree of variability at high frequencies under the flexible and constant money growth rules that gets smoothed out by the estimated interest rate rule. The graphs in figure 11 and the statistics in table 5 confirm, however, that the short-term interest rate actually has a *lower* standard deviation under the money growth rules than it did, historically, under the estimated interest rate rule.

Results from counterfactual simulations in Belongia and Ireland (2018) suggest that the

[17]Since inflation is observed in the estimated model, its posterior distribution puts all of its weight on the historical path.

US economy would have recovered more quickly even under a constant money growth rule. There are two explanations for the differences between those previous results and ones presented here. First, Belongia and Ireland (2018) use a structural vector autoregressive time series model with more flexible dynamics that allow changes in money growth to have effects on output and inflation even after controlling for movements in the short-term nominal interest rate. Thus, in the VAR, stability in money growth contributes to stability in output and inflation as well. Here, the New Keynesian model puts constant money growth rules at a disadvantage, by assuming that monetary policy actions have an impact on output and inflation exclusively through their effects on the current and expected future path of interest rates. To the extent that changes in money growth do play a separate role in the monetary transmission mechanism, as suggested by the empirical results in Belongia and Ireland (2018, 2021), the case for money growth rules grows stronger. Second, the simulations in Belongia and Ireland (2018) hold money growth constant during and after 2008, but at rates that are higher than the full-sample historical average. Therefore, the policy rules considered previously share with the flexible money growth rule (30) considered here the implication that money growth should respond, countercyclically, to movements in the output gap.

Finally, the second set of columns in the bottom panel of table 5 and the right-hand columns in figures 11 and 12 describe the outcome of another policy experiment: allowing the Fed to continue following the estimated Taylor rule (14), without monetary policy shocks, instead of resorting to forward guidance with zero nominal rates over the 2009-15 period. Under this counterfactual, the short-term nominal interest rate falls to a low of -2.30 percent in 2009:3. This additional monetary stimulus cushions the output gap somewhat during the recession itself. On the other hand, the negative interest rates prescribed by (14) fail to generate a more rapid recovery, or raise the inflation rate noticeably, relative to the Fed's historical policy of zero interest rates coupled with forward guidance.

Overall, therefore, results from table 5 and figures 11 and 12 send two messages. First, during 2009-15, the Fed's forward guidance *did* succeed in substituting for negative interest rates, in the sense that macroeconomic outcomes appear similar under these two scenarios. But, second, the flexible money growth rule (30), by supporting a much faster recovery without deflation, produces outcomes better than those under either the actual policy of forward guidance or the counterfactual of negative interest rates prescribed by (14). And the flexible money growth rate accomplishes this goal without requiring an extended period of negative nominal rates.

6 Conclusion

Historically, the Federal Reserve has conducted monetary policy by managing interest rates. New Keynesian models like the one estimated here justify this approach. Within these models, simple interest rate rules of the form proposed by Taylor (1993) capture well the Fed's actual interest rate decisions, work effectively to insulate the economy from the effects of money demand disturbances, and help the economy adjust more efficiently to other, non-monetary shocks. In these same models, constant money growth rules like those proposed by Friedman (1968) perform quite poorly. Consistent with Poole's (1970) more traditional Keynesian analysis, holding money growth constant in New Keynesian models allows money demand shocks to contribute significantly to additional macroeconomic volatility. Constant money growth rules fail, as well, to deliver the appropriate monetary response to other disturbances that require output and inflation to adjust.

The limitations of interest rate rules, however, are highlighted by more recent experience during and since the financial crisis and Great Recession of 2007-9. After lowering its federal funds rate target to a range close to zero, standard approaches to interest rate management were viewed to be impotent. This led the Fed to employ other, less conventional strategies such as forward guidance about the future interest rate path and large scale purchases of longer-term Treasury bonds and mortgage-backed securities in an attempt to deliver additional monetary stimulus and help the economy recover. These strategies led to a five-fold increase the monetary base that remains to be unwound, and also required the introduction of interest payments on reserves, a practice which must continue until the Fed's balance sheet reverts to pre-crisis levels.

This paper suggests an alternative approach that would have avoided these complications, and works well in good times and in bad. Departing from the strict confines of Friedman's constant money growth rule, it identifies a simple but slightly more flexible alternative that, like the Taylor rule, adjusts the rate of money growth, modestly and gradually, in response to movements in the output gap. Even without a direct response to money demand shocks, this rule helps the central bank accommodate those disturbances and, more generally, allows monetary policy to pursue short-run stabilization objectives even as it maintains an environment of nominal stability through its choice of the long-run money growth rate.

Counterfactual simulations reveal that this flexible money growth rule would have produced macroeconomic stability over the 1983:1-2019:4 sample period comparable to that observed, historically, under the estimated interest rate rule. Moreover, by targeting the rate of money growth and allowing interest rates to adjust, as needed, to maintain equilibrium in the market for bonds, the simulations show that this rule would have generated a more rapid recovery in both output and inflation after 2009, without resorting to forward guidance and with exceptionally low interest rates prevailing for less than one year.

These beneficial effects appear even in a standard New Keynesian model in which, by assumption, monetary policy actions are transmitted to the economy through their impact on interest rates and the stability of the money growth rate itself offers no additional advantage. To the extent that other channels of monetary transmission, like those identified empirically by Belongia and Ireland (2018, 2021), operate in the US economy, policy rules focusing on money growth may offer further advantages not captured here. Of course, implementing policy through a rule like the one proposed here might require institutional changes, such as the re-imposition of reserve requirements, that enhance the central bank's control over the broad monetary aggregates. The Fed's experiments with forward guidance, large-scale asset purchases, interest on reserves, and reverse repurchase agreements show, however, that institutional arrangements can be changed, dramatically and even on short notice, whenever circumstances demand more from monetary policy. In the wake of these experiments, shifting away from interest rate management and towards targeting money growth instead offers a simpler alternative, worthy of more serious consideration.

As noted above, the small-scale model used here provides an interpretation of the recent data that is broadly consistent with those offered by the larger models Campbell, Fisher, Justiniano, and Melosi (2016), Gust, Herbst, López-Salido, and Smith (2017), and Kulish, Morley, and Robinson (2017). At the same time, however, this small-scale model allows for the cleanest reconsideration of money growth rules in a setting where, as in Poole's (1970) classic analysis and Ireland (2000), Collard and Dellas (2005), and Galí's (2015) earlier extensions, the critical challenge is to prevent money demand disturbances from contributing to overall macroeconomic volatility.

It would, nevertheless, be interesting to explore the performance of money growth rules in more detailed models that account for physical capital accumulation, investment, and a wider range of structural shocks. With one such model, Campbell, Fisher, Justiniano, and Melosi (2016) show that replicating the co-movement between consumption and investment, particularly during and after the Great Recession of 2007-9, requires shocks to risk premia driven by underlying shifts in households' demand for safe and liquid bonds. After incorporating these additional elements, the model considered here would become one in which *both* money and bonds appear in the utility function. A host of additional questions regarding the optimal management of the supplies of money and government bonds – perhaps through coordinated monetary and fiscal policies at the zero lower interest rate bound – would arise and could be considered. This ambitious extension is left for future research.

7 References

Abo-Zaid, Salem and Julio Garín. "Optimal Monetary Policy and Imperfect Financial Markets: A Case for Negative Nominal Interest Rates?" *Economic Inquiry* 54 (January 2016): 215-228.

Adjemian, Stéphane. "Prior Distributions in Dynare." Manuscript: Université du Maine, April 2016.

Anderson, Brian D.O. and John B. Moore. *Optimal Filtering.* Mineola: Dover Publications, 2005.

Anderson, Richard G., Michael Bordo, and John V. Duca. "Money and Velocity During Financial Crises: From the Great Depression to the Great Recession." *Journal of Economic Dynamics and Control* 81 (August 2017): 32-49.

Andrés, Javier, J., David López-Salido, and Edward Nelson. "Tobin's Imperfect Asset Substitution in Optimizing General Equilibrium." *Journal of Money, Credit, and Banking* 36 (August 2004): 665-690.

Andrés, Javier, J., David López-Salido, and Edward Nelson. "Money and the Natural Rate of Interest: Structural Estimates for the United States and Euro Area." *Journal of Economic Dynamics and Control* 33 (March 2009): 758-776.

Aruoba, S. Borağan, Pablo Cuba-Borda, and Frank Schorfheide. "Macroeconomic Dynamics Near the ZLB: A Tale of Two Countries." *Review of Economic Studies* 85 (January 2018): 87-118.

Barnett, William A. *Getting It Wrong: How Faulty Monetary Statistics Undermine the Fed, the Financial System, and the Economy.* Cambridge: MIT Press, 2012.

Barnett, William A. and Marcelle Chauvet. "How Better Monetary Statistics Could Have Signaled the Financial Crisis." *Journal of Econometrics* 161 (March 2011): 6-23.

Barnett, William A., Jia Liu, Ryan S. Mattson, and Jeff van den Noort. "The New CFS Divisia Monetary Aggregates: Design, Construction, and Data Sources." *Open Economies Review* 24 (February 2013): 101-124.

Beck, Guenter W. and Volker Wieland. "Central Bank Misperceptions and the Role of Money in Interest-Rate Rules." *Journal of Monetary Economics* 55 (October 2008): S1-S17.

Belongia, Michael T. "Measurement Matters: Recent Results in Monetary Economics Re-examined." *Journal of Political Economy* 104 (October 1996): 1065-1083.

Belongia, Michael T. and Peter N. Ireland. "Interest Rates and Money in the Measurement of Monetary Policy." *Journal of Business and Economic Statistics* 33 (April 2015*a*): 255-269.

Belongia, Michael T. and Peter N. Ireland. "A 'Working' Solution to the Question of Nominal GDP Targeting." *Macroeconomic Dynamics* 19 (April 2015*b*): 508-534.

Belongia, Michael T. and Peter N. Ireland. "Money and Output: Friedman and Schwartz Revisited." *Journal of Money, Credit, and Banking* 48 (September 2016): 1223-1266.

Belongia, Michael T. and Peter N. Ireland. "Circumventing the Zero Lower Bound with Monetary Policy Rules Based on Money." *Journal of Macroeconomics* 54 (December 2017, Part A): 42-58.

Belongia, Michael T. and Peter N. Ireland. "Targeting Constant Money Growth at the Zero Lower Bound." *International Journal of Central Banking* 14 (March 2018): 159-204.

Belongia, Michael T. and Peter N. Ireland. "The Demand for Divisia Money: Theory and Evidence." *Journal of Macroeconomics* 61 (September 2019): Article 103128.

Belongia, Michael T. and Peter N. Ireland. "A Classical View of the Business Cycle." *Journal of Money, Credit, and Banking* 53 (March-April 2021): 333-366.

Bianchi, Francesco and Leonardo Melosi. "Escaping the Great Recession." *American Economic Review* 107 (April 2017): 1030-1058.

Billi, Roberto M., Ulf Söderström, and Carl E. Walsh. "The Role of Money in Monetary Policy at the Lower Bound." Manuscript. Stockholm: Sveriges Riksbank, June 2021.

Campbell, Jeffrey R., Jonas D.M. Fisher, Alejandro Justiniano, and Leonardo Melosi. "Forward Guidance and Macroeconomic Outcomes since the Financial Crisis." *NBER Macroeconomics Annual* 31 (2016): 283-357.

Carlstrom, Charles T., Timothy S. Fuerst, and Matthias Paustian. "Inflation and Output in New Keynesian Models with a Transient Interest Rate Peg." *Journal of Monetary Economics* 76 (November 2015): 230-243.

Chib, Siddhartha and Srikanth Ramamurthy. "Tailored Randomized Block MCMC Methods with Applications to DSGE Models." *Journal of Econometrics* 155 (March 2010): 19-38.

Chow, Gregory C. "On the Long-Run and Short-Run Demand for Money." *Journal of Political Economy* 74 (April 1966): 111-131.

Clarida, Richard, Jordi Galí, and Mark Gertler. "The Science of Monetary Policy: A New Keynesian Perspective." *Journal of Economic Literature* 37 (December 1999): 1661-1707.

Collard, Fabrice and Harris Dellas. "Poole in the New Keynesian Model." *European Economic Review* 49 (May 2005): 887-907.

Cook, Timothy. "Determinants of the Federal Funds Rate: 1979-1982." Federal Reserve Bank of Richmond *Economic Review* (January/February 1989): 3-19.

Del Negro, Marco, Marc P. Giannoni, and Christina Patterson. "The Forward Guidance Puzzle." Staff Report 574. New York: Federal Reserve Bank of New York, December

2015.

Del Negro, Marco, Marc P. Giannoni, and Frank Schorfheide. "Inflation in the Great Recession and New Keynesian Models." *American Economic Journal: Macroeconmomics* 7 (January 2015): 168-196.

Dong, Feng and Yi Wen. "Optimal Monetary Policy under Negative Interest Rate." Working Paper 2017-019A. St. Louis: Federal Reserve Bank of St. Louis, May 2017.

Duca, John V. "Financial Technology Shocks and the Case of the Missing M2." *Journal of Money, Credit, and Banking* 32 (November 2000, Part 1): 820-839.

Durbin, J. and S.J. Koopman. "A Simple and Efficient Simulation Smoother for State Space Time Series Analysis." *Biometrika* 89 (September 2002): 603-615.

Edge, Rochelle M. and Refet S. Gurkaynak. "How Useful Are Estimated DSGE Model Forecasts for Central Bankers?" *Brookings Papers on Economic Activity* (Fall 2010): 209-244.

Ellington, Michael. "The Case for Divisia Monetary Statistics: A Bayesian Time-Varying Approach." *Journal of Economic Dynamics and Control* 96 (November 2018): 26-41.

Fernández-Villaverde, Jesús, Grey Gordon, Pablo Guerrón-Quintana, and Juan F. Rubio-Ramírez. "Nonlinear Adventures at the Zero Lower Bound." *Journal of Economic Dynamics and Control* 57 (August 2015): 182-204.

Friedman, Milton. "The Role of Monetary Policy." *American Economic Review* 58 (March 1968): 1-17.

Galí, Jordi. *Monetary Policy, Inflation, and the Business Cycle*, 2nd Ed. Princeton: Princeton University Press, 2015.

Geweke, John. "Using Simulation Methods for Bayesian Econometric Models: Inference, Development, and Communication." *Econometric Reviews* 18 (1999, Issue 1): 1-73.

Gilbert, R. Alton. "A Case Study in Monetary Control: 1980-82." Federal Reserve Bank of St. Louis *Review* 76 (September/October 1994): 35-58.

Goldfeld, Stephen M. "The Demand for Money Revisited." *Brookings Papers on Economic Activity* (1973, Issue 3): 577-638.

Greenspan, Alan. "Remarks by Chairman Alan Greenspan: Rules vs. Discretionary Monetary Policy." Speech at the 15th Anniversary Conference of the Center for Economic Policy Research. Stanford: Stanford University, 5 September 1997.

Guerrieri, Luca and Matteo Iacoviello. "OccBin: A Toolkit for Solving Dynamic Models with Occasionally Binding Constraints Easily." *Journal of Monetary Economics* 70 (March 2015): 22-38.

Gust, Christopher, Edward Herbst, David López-Salido, and Matthew E. Smith. "The Empirical Implications of the Interest-Rate Lower Bound." *American Economic Review* 107 (July 2017): 1971-2006.

Hallman, Jeffrey J., Richard D. Porter, and David H. Small. "Is the Price Level Tied to the M2 Monetary Aggregate in the Long-Run?" *American Economic Review* 81 (September 1991): 841-858.

Harrison, Richard. "Estimating the Effects of Forward Guidance in Rational Expectations Models." *European Economic Review* 79 (October 2015): 196-213.

Harvey, Andrew C. *Forecasting, Structural Time Series Models and the Kalman Filter.* Cambridge: Cambridge University Press, 1989.

Hendrickson, Joshua R. "Redundancy or Mismeasurement? A Reappraisal of Money." *Macroeconomic Dynamics* 18 (October 2014): 1437-1465.

Ireland, Peter N. "The Role of Countercyclical Monetary Policy." *Journal of Political Economy* 104 (August 1996): 704-723.

Ireland, Peter N. "Interest Rates, Inflation, and Federal Reserve Policy Since 1980." *Journal of Money, Credit, and Banking* 32 (August 2000, Part 1): 417-434.

Ireland, Peter N. "A Method for Taking Models to the Data." *Journal of Economic Dynamics and Control* 28 (March 2004*a*): 1205-1226.

Ireland, Peter N. "Technology Shocks in the New Keynesian Model." *Review of Economics and Statistics* 86 (November 2004*b*): 923-936.

Ireland, Peter N. "Money's Role in the Monetary Business Cycle." *Journal of Money, Credit, and Banking* 36 (December 2004*c*): 969-983.

Ireland, Peter N. "Changes in the Federal Reserve's Inflation Target: Causes and Consequences." *Journal of Money, Credit, and Banking* 39 (December 2007): 1851-1882.

Ireland, Peter N. "On the Welfare Cost of Inflation and the Recent Behavior of Money Demand." *American Economic Review* 99 (June 2009): 1040-1052.

Ireland, Peter N. "A New Keynesian Perspective on the Great Recession." *Journal of Money, Credit, and Banking* 43 (February 2011): 31-54.

Jackson, Harriet. "The International Experience with Negative Policy Rates." Staff Discussion Paper 2015-13. Ottawa: Bank of Canada, November 2015.

Judson, Ruth A., Bernd Schlusche, and Vivian Wong. "Demand for M2 at the Zero Lower Bound: The Recent U.S. Experience." Finance and Economics Discussion Series 2014-22. Washington: Federal Reserve Board, January 2014.

Keating, John W. , Logan J. Kelly, A. Lee Smith, and Victor J. Valcarcel. "A Model of Monetary Policy Shocks for Financial Crises and Normal Conditions." *Journal of Money, Credit, and Banking* 51 (February 2019): 227-259.

King, Robert G., Charles I. Plosser, and Sergio T. Rebelo. "Production, Growth and Business Cycles: I. The Basic Neoclassical Model." *Journal of Monetary Economics* 21 (March-May 1988): 195-232.

Koop, Gary, M. Hashem Pesaran, and Simon M. Potter. "Impulse Response Analysis in Nonlinear Multivariate Models." *Journal of Econometrics* 74 (September 1996): 119-147.

Kulish, Mariano, James Morley, and Tim Robinson. "Estimating DSGE Models with Zero Interest Rate Policy." *Journal of Monetary Economics* 88 (June 2017): 35-49.

Meulendyke, Ann-Marie. *U.S. Monetary Policy and Financial Markets.* New York: Federal Reserve Bank of New York, 1998.

Nelson, Edward. "Direct Effects of Base Money on Aggregate Demand: Theory and Evidence." *Journal of Monetary Economics* 49 (May 2002): 687-708.

Orphanides, Athanasios and Richard D. Porter. "P* Revisited: Money-Based Inflation Forecasts with a Changing Equilibrium Velocity." *Journal of Economics and Business* 52 (January-April 2000): 87-100.

Poole, William. "Optimal Choice of Monetary Policy Instruments in a Simple Stochasic Macro Model." *Quarterly Journal of Economics* 84 (May 1970): 197-216.

Ríos-Rull, José-Víctor, Frank Schorfheide, Cristina Fuentes-Albero, Maxym Kryshko, and Raül Santaeulàlia-Llopis. "Methods versus Substance: Measuring the Effects of Technology Shocks." *Journal of Monetary Economics* 59 (December 2012): 826-846.

Rognlie, Matthew. "What Lower Bound? Monetary Policy with Negative Interest Rates." Manuscript. Evanston: Northwestern University, July 2016.

Rotemberg, Julio J. "Sticky Prices in the United States." *Journal of Political Economy* 90 (December 1982): 1187-1211.

Serletis, Apostolos and Periklis Gogas. "Divisia Monetary Aggregates, the Great Ratios, and Classical Money Demand Functions." *Journal of Money, Credit, and Banking* 46 (February 2014): 229-241.

Smets, Frank and Rafael Wouters. "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach." *American Economic Review* 97 (June 2007): 586-606.

Swanson, Eric T. and John C. Williams. "Measuring the Effect of the Zero Lower Bound on Medium- and Longer-Term Interest Rates." *American Economic Review* 104 (October 2014): 3154-3185.

Taylor, John B. "Discretion Versus Policy Rules in Practice." *Carnegie-Rochester Conference Series on Public Policy* 39 (December 1993): 195-124.

Taylor, John B. *Monetary Policy Rules.* Chicago: University of Chicago Press, 1999.

Thornton, Daniel L. "When Did the FOMC Begin Targeting the Federal Funds Rate? What the Verbatim Transcripts Tell Us." *Journal of Money, Credit, and Banking* 38 (December 2006): 2039-2071.

Woodford, Michael. *Interest and Prices: Foundations of a Theory of Monetary Policy.* Princeton: Princeton University Press, 2003.

8 Appendix

8.1 Optimality Conditions

The representative household chooses C_t, h_t, B_t, and M_t for all $t = 0, 1, 2, \ldots$ to maximize expected utility (2) subject to the budget constraint (1) for all $t = 0, 1, 2, \ldots$. The first-order conditions for this problem can be written as

$$\Lambda_t = \frac{a_t}{C_t - \gamma C_{t-1}} - \beta\gamma E_t\left(\frac{a_{t+1}}{C_{t+1} - \gamma C_t}\right), \tag{A.1}$$

$$\chi_0 a_t h_t^{\chi} = \Lambda_t (W_t/P_t), \tag{A.2}$$

$$\Lambda_t = \beta r_t E_t(\Lambda_{t+1}/\pi_{t+1}), \tag{A.3}$$

$$\begin{aligned} & a_t v_1\left(\frac{M_t}{P_t Z_t}, u_t\right) - a_t\left(\frac{\phi_m}{2}\right)\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right)^2 \\ & - a_t\phi_m\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right)\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}}\right) \\ & + \beta\phi_m E_t\left[a_{t+1}\left(\frac{M_{t+1}/P_{t+1}}{zM_t/P_t} - 1\right)\left(\frac{M_{t+1}/P_{t+1}}{zM_t/P_t}\right)^2\left(\frac{zZ_t}{Z_{t+1}}\right)\right] \\ = {} & Z_t\Lambda_t\left(1 - \frac{1}{r_t}\right), \end{aligned} \tag{A.4}$$

and (1) with equality for all $t = 0, 1, 2, \ldots$, where Λ_t denotes the nonnegative Lagrange multiplier on the budget constraint for period t.

In the special case where $\gamma = 0$ and $\phi_m = 0$, (A.1) can be substituted into (A.4) to obtain the model's long-run money demand specification (6). More generally, when the utility function for real balances takes the form introduced in (7), (A.4) becomes

$$\begin{aligned} & \frac{a_t}{\delta}\left[\ln(m^*) - \ln\left(\frac{M_t}{P_t Z_t}\right) + \ln(u_t)\right] - a_t\left(\frac{\phi_m}{2}\right)\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right)^2 \\ & - a_t\phi_m\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right)\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}}\right) \\ & + \beta\phi_m E_t\left[a_{t+1}\left(\frac{M_{t+1}/P_{t+1}}{zM_t/P_t} - 1\right)\left(\frac{M_{t+1}/P_{t+1}}{zM_t/P_t}\right)^2\left(\frac{zZ_t}{Z_{t+1}}\right)\right] \\ = {} & Z_t\Lambda_t\left(1 - \frac{1}{r_t}\right). \end{aligned} \tag{A.5}$$

During each period t, the representative finished goods-producing firm chooses $Y_t(i)$ for all $i \in [0, 1]$ to maximize its profits, which are given by

$$P_t\left[\int_0^1 Y_t(i)^{(\theta_t-1)/\theta_t}\,\mathrm{d}i\right]^{\theta_t/(\theta_t-1)} - \int_0^1 P_t(i)Y_t(i)\,\mathrm{d}i.$$

The first-order conditions for this problem are

$$Y_t(i) = [P_t(i)/P_t]^{-\theta_t} Y_t$$

for all $i \in [0,1]$ and $t = 0,1,2,\ldots$. Competition drives the finished goods-producing firm's profits to zero in equilibrium, determining P_t as

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta_t}\,\mathrm{d}i\right]^{1/(1-\theta_t)}$$

for all $t = 0,1,2,\ldots$.

Meanwhile, the cost of price adjustment makes the representative intermediate goods-producing firm's problem dynamic: it chooses $P_t(i)$ for all $t = 0,1,2,\ldots$ to maximize its total real market value, proportional to

$$E_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t [D_t(i)/P_t],$$

where $\beta^t \Lambda_t$ measures the marginal utility value to the representative household of an additional unit of real profits received in the form of dividends during period t and where

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t}\right]^{1-\theta_t} Y_t - \left[\frac{P_t(i)}{P_t}\right]^{-\theta_t} \left(\frac{W_t}{P_t}\right)\left(\frac{Y_t}{Z_t}\right) - \frac{\phi}{2}\left[\frac{P_t(i)}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}P_{t-1}(i)} - 1\right]^2 Y_t \quad \text{(A.6)}$$

measures the firm's real profits during the same period t. The first-order conditions for this problem are

$$\begin{aligned} 0 = {} & (1-\theta_t)\left[\frac{P_t(i)}{P_t}\right]^{-\theta_t} + \theta_t \left[\frac{P_t(i)}{P_t}\right]^{-\theta_t - 1}\left(\frac{W_t}{P_t}\right)\left(\frac{1}{Z_t}\right) \\ & - \phi_p \left[\frac{P_t(i)}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}P_{t-1}(i)} - 1\right]\left[\frac{P_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}P_{t-1}(i)}\right] \\ & + \beta\phi_p E_t \left\{\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)\left[\frac{P_{t+1}(i)}{\pi_t^{\alpha}\pi^{1-\alpha}P_t(i)} - 1\right]\left[\frac{P_{t+1}(i)}{\pi_t^{\alpha}\pi^{1-\alpha}P_t(i)}\right]\left[\frac{P_t}{P_t(i)}\right]\left(\frac{Y_{t+1}}{Y_t}\right)\right\} \end{aligned} \quad \text{(A.7)}$$

for all $t = 0,1,2,\ldots$.

Finally, the social planner chooses Q_t and $n_t(i)$ for all $i \in [0,1]$ to maximize the social welfare function in (11) subject to the aggregate feasibility constraint (12) for all $t = 0,1,2,\ldots$. The first-order conditions for this problem are

$$\Xi_t = \frac{a_t}{Q_t - \gamma Q_{t-1}} - \beta\gamma E_t\left(\frac{a_{t+1}}{Q_{t+1} - \gamma Q_t}\right),$$

$$a_t \chi_0 \left[\int_0^1 n_t(i)\,\mathrm{d}i\right]^{\chi} = \Xi_t Z_t (Q_t/Z_t)^{1/\theta_t} n_t(i)^{-1/\theta_t}$$

for all $i \in [0,1]$, and (12) with equality for all $t = 0,1,2,\ldots$, where Ξ_t denotes the nonnegative

Lagrange multiplier on the aggregate feasibility constraint for period t.

The second optimality condition listed above implies that $n_t(i) = n_t$ for all $i \in [0,1]$ and $t = 0, 1, 2, \ldots$, where

$$n_t = (\Xi_t/\chi_0)^{\theta_t/(1+\chi\theta_t)}(Z_t/a_t)^{\theta_t/(1+\chi\theta_t)}(Q_t/Z_t)^{1/(1+\chi\theta_t)}$$

Substituting this last relationship into (12) yields

$$\Xi_t = \chi_0(a_t/Z_t)(Q_t/Z_t)^{\chi}.$$

Hence, the efficient level of output must satisfy

$$\left(\frac{\chi_0}{Z_t}\right)\left(\frac{Q_t}{Z_t}\right)^{\chi} = \frac{1}{Q_t - \gamma Q_{t-1}} - \beta\gamma E_t\left[\left(\frac{a_{t+1}}{a_t}\right)\left(\frac{1}{Q_{t+1} - \gamma Q_t}\right)\right] \tag{A.8}$$

for all $t = 0, 1, 2, \ldots$.

8.2 Deriving the Log-Linearized Model

After imposing the symmetry and market clearing conditions $Y_t(i) = Y_t$, $h_t(i) = h_t$, $D_t(i) = D_t$, and $P_t(i) = P_t$ for all $i \in [0,1]$ and $t = 0, 1, 2, \ldots$ and $M_t = M_{t-1} + T_t$ and $B_t = B_{t-1} = 0$ for all $t = 0, 1, 2, \ldots$, (10), (A.2), and (A.6) can be used to solve out for W_t/P_t, h_t, and D_t. The system implied by (1), (3)-(5), (9), (13)-(16), (A.1), (A.3), (A.5), (A.7), and (A.8) then becomes

$$Y_t = C_t + \frac{\phi_p}{2}\left(\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}} - 1\right)^2 Y_t, \tag{1}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{3}$$

$$\ln(Z_t) = \ln(z) + \ln(Z_{t-1}) + \varepsilon_{zt}, \tag{4}$$

$$\ln(u_t) = \rho_u \ln(u_{t-1}) + \varepsilon_{ut}, \tag{5}$$

$$\ln(\theta_t) = (1-\rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \varepsilon_{\theta t}, \tag{9}$$

$$x_t = Y_t/Q_t, \tag{13}$$

$$\ln(r_t/r) = \rho_r \ln(r_{t-1}/r) + \rho_\pi \ln(\pi_t/\pi) + \rho_x \ln(x_t/x) + \varepsilon_{rt}, \tag{14}$$

$$\mu_t = \left(\frac{M_t/P_t}{M_{t-1}/P_{t-1}}\right)\pi_t, \tag{15}$$

$$g_t = Y_t/Y_{t-1}, \tag{16}$$

$$\Lambda_t = \frac{a_t}{C_t - \gamma C_{t-1}} - \beta\gamma E_t\left(\frac{a_{t+1}}{C_{t+1} - \gamma C_t}\right), \tag{A.1}$$

$$\Lambda_t = \beta r_t E_t(\Lambda_{t+1}/\pi_{t+1}), \tag{A.3}$$

$$
\begin{aligned}
&\frac{a_t}{\delta}\left[\ln(m^*) - \ln\left(\frac{M_t}{P_t Z_t}\right) + \ln(u_t)\right] - a_t\left(\frac{\phi_m}{2}\right)\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right)^2 \\
&- a_t\phi_m\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right)\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}}\right) \\
&+ \beta\phi_m E_t\left[a_{t+1}\left(\frac{M_{t+1}/P_{t+1}}{zM_t/P_t} - 1\right)\left(\frac{M_{t+1}/P_{t+1}}{zM_t/P_t}\right)^2\left(\frac{zZ_t}{Z_{t+1}}\right)\right] \\
= {} & Z_t\Lambda_t\left(1 - \frac{1}{r_t}\right),
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
\theta_t - 1 = {} & \theta_t\left(\frac{\chi_0 a_t}{\Lambda_t Z_t}\right)\left(\frac{Y_t}{Z_t}\right)^\chi - \phi_p\left(\frac{\pi_t}{\pi_{t-1}^\alpha \pi^{1-\alpha}} - 1\right)\left(\frac{\pi_t}{\pi_{t-1}^\alpha \pi^{1-\alpha}}\right) \\
&+ \beta\phi_p E_t\left[\left(\frac{\Lambda_{t+1}Y_{t+1}}{\Lambda_t Y_t}\right)\left(\frac{\pi_{t+1}}{\pi_t^\alpha \pi^{1-\alpha}} - 1\right)\left(\frac{\pi_{t+1}}{\pi_t^\alpha \pi^{1-\alpha}}\right)\right],
\end{aligned}
\tag{A.7}
$$

and

$$
\left(\frac{\chi_0}{Z_t}\right)\left(\frac{Q_t}{Z_t}\right)^\chi = \frac{1}{Q_t - \gamma Q_{t-1}} - \beta\gamma E_t\left[\left(\frac{a_{t+1}}{a_t}\right)\left(\frac{1}{Q_{t+1} - \gamma Q_t}\right)\right] \tag{A.8}
$$

for all $t = 0, 1, 2, \ldots$.

In terms of the stationary variables $y_t = Y_t/Z_t$, $c_t = C_t/Z_t$, π_t, r_t, $m_t = (M_t/P_t)/Z_t$, $q_t = Q_t/Z_t$, x_t, μ_t, g_t, $\lambda_t = Z_t\Lambda_t$, a_t, $z_t = Z_t/Z_{t-1}$, u_t, and θ_t, the system of symmetric equilibrium conditions can be rewritten as

$$
y_t = c_t + \frac{\phi_p}{2}\left(\frac{\pi_t}{\pi_{t-1}^\alpha \pi^{1-\alpha}} - 1\right)^2 y_t, \tag{1}
$$

$$
\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{3}
$$

$$
\ln(z_t) = \ln(z) + \varepsilon_{zt}, \tag{4}
$$

$$
\ln(u_t) = \rho_u \ln(u_{t-1}) + \varepsilon_{ut}, \tag{5}
$$

$$
\ln(\theta_t) = (1-\rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \varepsilon_{\theta t}, \tag{9}
$$

$$
x_t = y_t/q_t, \tag{13}
$$

$$
\ln(r_t/r) = \rho_r \ln(r_{t-1}/r) + \rho_\pi \ln(\pi_t/\pi) + \rho_x \ln(x_t/x) + \varepsilon_{rt}, \tag{14}
$$

$$
\mu_t = z_t(m_t/m_{t-1})\pi_t, \tag{15}
$$

$$
g_t = (y_t/y_{t-1})z_t, \tag{16}
$$

$$
\lambda_t = \frac{a_t z_t}{z_t c_t - \gamma c_{t-1}} - \beta\gamma E_t\left(\frac{a_{t+1}}{z_{t+1}c_{t+1} - \gamma c_t}\right), \tag{A.1}
$$

$$
\lambda_t = \beta r_t E_t\left(\frac{\lambda_{t+1}}{z_{t+1}\pi_{t+1}}\right), \tag{A.3}
$$

$$\begin{aligned}
&\frac{a_t}{\delta}[\ln(m^*) - \ln(m_t) + \ln(u_t)] - a_t\left(\frac{\phi_m}{2}\right)\left(\frac{z_t m_t}{z m_{t-1}} - 1\right)^2 \\
&- a_t\phi_m\left(\frac{z_t m_t}{z m_{t-1}} - 1\right)\left(\frac{z_t m_t}{z m_{t-1}}\right) \\
&+ \beta\phi_m E_t\left[a_{t+1}\left(\frac{z_{t+1} m_{t+1}}{z m_t} - 1\right)\left(\frac{z_{t+1} m_{t+1}}{z m_t}\right)^2\left(\frac{z}{z_{t+1}}\right)\right] \\
= {}&\lambda_t\left(1 - \frac{1}{r_t}\right),
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
\theta_t - 1 = {}&\theta_t\chi_0\left(\frac{a_t}{\lambda_t}\right)y_t^\chi - \phi_p\left(\frac{\pi_t}{\pi_{t-1}^\alpha\pi^{1-\alpha}} - 1\right)\left(\frac{\pi_t}{\pi_{t-1}^\alpha\pi^{1-\alpha}}\right) \\
&+ \beta\phi_p E_t\left[\left(\frac{\lambda_{t+1}y_{t+1}}{\lambda_t y_t}\right)\left(\frac{\pi_{t+1}}{\pi_t^\alpha\pi^{1-\alpha}} - 1\right)\left(\frac{\pi_{t+1}}{\pi_t^\alpha\pi^{1-\alpha}}\right)\right],
\end{aligned} \tag{A.7}$$

and

$$\chi_0 q_t^\chi = \frac{z_t}{z_t q_t - \gamma q_{t-1}} - \beta\gamma E_t\left[\left(\frac{a_{t+1}}{a_t}\right)\left(\frac{1}{z_{t+1}q_{t+1} - \gamma q_t}\right)\right] \tag{A.8}$$

for all $t = 0, 1, 2, \ldots$.

The stationary system pins down the steady-state values $y_t = y$, $c_t = c$, $\pi_t = \pi$, $r_t = r$, $m_t = m$, $q_t = q$, $x_t = x$, $\mu_t = \mu$, $g_t = g$, $\lambda_t = \lambda$, $a_t = a = 1$, $z_t = z$, $u_t = u = 1$, and $\theta_t = \theta$. In particular, $a = 1$, z, $u = 1$, and θ are determined exogenously by (3)-(5) and (9), and π is chosen by the central bank through (14). Equations (1), (15), and (16) then imply that $c = y$, $\mu = z\pi$, and $g = z$.

Equations (A.1) and (A.7) imply that

$$y = \left[\left(\frac{1}{\chi_0}\right)\left(\frac{\theta - 1}{\theta}\right)\left(\frac{z - \beta\gamma}{z - \gamma}\right)\right]^{1/(1+\chi)}$$

and

$$\lambda = \chi_0\left(\frac{\theta}{\theta - 1}\right)y^\chi,$$

while (A.8) implies that

$$q = \left[\left(\frac{1}{\chi_0}\right)\left(\frac{z - \beta\gamma}{z - \gamma}\right)\right]^{1/(1+\chi)}$$

so that (13) implies

$$x = \left(\frac{\theta - 1}{\theta}\right)^{1/(1+\chi)}.$$

Equation (A.3) implies

$$r = (z/\beta)\pi.$$

Finally, (A.5) implies

$$\ln(m) = \ln(m^*) - \delta_r(r - 1),$$

where δ_r is given by (23).

The system consisting of (1), (3)-(5), (9), (13)-(16), (A.1), (A.3), (A.5), (A.7), and (A.8) can also be log-linearized around the steady-state to describe how the economy responds to shocks. Let $\hat{y}_t = \ln(y_t/y)$, $\hat{c}_t = \ln(c_t/c)$, $\hat{\pi}_t = \ln(\pi_t/\pi)$, $\hat{r}_t = \ln(r_t/r)$, $\hat{m}_t = \ln(m_t/m)$, $\hat{q}_t = \ln(q_t/q)$, $\hat{x}_t = \ln(x_t/x)$, $\hat{\mu}_t = \ln(\mu_t/\mu)$, $\hat{g}_t = \ln(g_t/g)$, $\hat{\lambda}_t = \ln(\lambda_t/\lambda)$, $\hat{a}_t = \ln(a_t)$, $\hat{z}_t = \ln(z_t/z)$, $\hat{u}_t = \ln(u_t)$, and $\hat{\theta}_t = \ln(\theta_t/\theta)$ denote the percentage deviation of each variable from its steady-state level. A first-order Taylor approximation to (1) implies that $\hat{c}_t = \hat{y}_t$. First-order approximations to the remaining 13 equations then imply

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_{at}, \tag{A.9}$$

$$\hat{z}_t = \varepsilon_{zt}, \tag{A.10}$$

$$\hat{u}_t = \rho_u \hat{u}_{t-1} + \varepsilon_{ut}, \tag{A.11}$$

$$\hat{e}_t = \rho_e \hat{e}_{t-1} + \varepsilon_{et}, \tag{A.12}$$

$$\hat{x}_t = \hat{y}_t - \hat{q}_t, \tag{A.13}$$

$$\hat{r}_t = \rho_r \hat{r}_{t-1} + \rho_\pi \hat{\pi}_t + \rho_x \hat{x}_t + \varepsilon_{rt}, \tag{A.14}$$

$$\hat{\mu}_t = \hat{z}_t + \hat{m}_t - \hat{m}_{t-1} + \hat{\pi}_t, \tag{A.15}$$

$$\hat{g}_t = \hat{y}_t - \hat{y}_{t-1} + \hat{z}_t, \tag{A.16}$$

$$(z-\beta\gamma)(z-\gamma)\hat{\lambda}_t = \gamma z \hat{y}_{t-1} - (z^2+\beta\gamma^2)\hat{y}_t + \beta\gamma z E_t \hat{y}_{t+1} + (z-\beta\gamma\rho_a)(z-\gamma)\hat{a}_t - \gamma z \hat{z}_t, \tag{A.17}$$

$$\hat{\lambda}_t = \hat{r}_t + E_t \hat{\lambda}_{t+1} - E_t \hat{\pi}_{t+1}, \tag{A.18}$$

$$\delta_r(r-1)\hat{\lambda}_t - \delta_r(r-1)\hat{a}_t - \hat{u}_t + \phi \hat{z}_t = \phi \hat{m}_{t-1} - [1+(1+\beta)\phi]\hat{m}_t + \beta\phi E_t \hat{m}_{t+1} - \delta_r \hat{r}_t, \tag{A.19}$$

$$(1+\beta\alpha)\hat{\pi}_t = \alpha\hat{\pi}_{t-1} + \beta E_t \hat{\pi}_{t+1} - \psi\hat{\lambda}_t + \psi\hat{a}_t + \chi\psi\hat{y}_t + \hat{e}_t, \tag{A.20}$$

and

$$0 = \gamma z \hat{q}_{t-1} - [z^2+\beta\gamma^2+\chi(z-\beta\gamma)(z-\gamma)]\hat{q}_t + \beta\gamma z E_t \hat{q}_{t+1} + \beta\gamma(z-\gamma)(1-\rho_a)\hat{a}_t - \gamma z \hat{z}_t \tag{A.21}$$

for all $t = 0, 1, 2, \ldots$. In (A.12) and (A.20), the cost push shock has been renormalized as $\hat{e}_t = -(1/\phi_p)\hat{\theta}_t$, so that ε_{et} is normally distributed with mean zero and standard deviation $\sigma_e = \sigma_\theta/\phi_p$. The persistence parameter $\rho_e = \rho_\theta$, and ψ is given by (21). Finally, in (A.19), the adjustment cost parameter for real balances is renormalized as in (29).

Equations (A.13), (A.17), (A.18), and (A.21), which are log-linearized versions of (13), (A.1), (A.3), and (A.8), define the model's New Keynesian IS relationship linking movements in the output gap $\hat{x}_t$ to the real interest rate $\hat{r}_t - E_t\hat{\pi}_{t+1}$, with backward-looking elements introduced through habit formation in the representative household's utility function. In the special case where $\gamma = 0$, so that habit formation is absent, these equations combine to yield the simpler, purely-forward looking specification in (19). Meanwhile (A.20), the linearized form of (A.7), is the New Keynesian Phillips curve, again with a backward-looking component entering when $\alpha > 0$, so that sticky individual goods prices are indexed to past inflation; when $\alpha = 0$, price-setting is purely forward-looking, as shown in (20). Together with the Taylor rule (A.14) for monetary policy, derived directly from (14), this first block of equations works, exactly as in textbook New Keynesian models, to determine the dynamic behavior of inflation, output and the output gap, and the short-term nominal interest rate without reference to the behavior of the money stock, whether real or nominal.

The addition of the money demand relationship (A.19) implied by (A.5), however, serves to determine the level of real balances in equilibrium under the Taylor rule (14). Focusing again on the special case where $\gamma = 0$ and $\phi_m = 0$, so that there is no habit formation in consumption or adjustment costs for real balances, (A.17) and (A.19) combine to yield the money demand equation (22). Finally, in this linearized system, (A.15) and (A.16) follow from (15) and (16) to determine the growth rate of the nominal money stock and aggregate output, and (A.9)-(A.12), which restate (3)-(5) and (9), govern the dynamics of the preference, productivity, money demand, and cost-push shocks.

8.3 Solving the Log-Linearized Model

Let

$$s_{0,t} = \begin{bmatrix} \hat{y}_t & \hat{\pi}_t & \hat{r}_t & \hat{m}_t & \hat{g}_t & \hat{\mu}_t & \hat{q}_t & \hat{x}_t & \hat{\lambda}_t \end{bmatrix}'$$

and

$$\xi_t = \begin{bmatrix} \hat{a}_t & \hat{z}_t & \hat{u}_t & \hat{e}_t & \varepsilon_{rt} \end{bmatrix}'.$$

Then (A.13)-(A.21) can be written as

$$A_0 s_{0,t} = A_1 s_{0,t-1} + B_0 E_t s_{0,t+1} + C_0 \xi_t, \tag{24}$$

where A_0, A_1, and B_0 are 9×9 matrices and C_0 is a 9×5 matrix, with elements determined by the model's structural parameters. After inverting A_0, (24) can be written in the slightly simpler form

$$s_{0,t} = A s_{0,t-1} + B E_t s_{0,t+1} + C \xi_t, \tag{A.22}$$

where $A = A_0^{-1} A_1$, $B = A_0^{-1} B_0$, and $C = A_0^{-1} C_0$. Meanwhile, (A.9)-(A.12) can be written as

$$\xi_t = P \xi_{t-1} + \varepsilon_t, \tag{25}$$

$$P = \begin{bmatrix} \rho_a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_u & 0 & 0 \\ 0 & 0 & 0 & \rho_e & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{at} & \varepsilon_{zt} & \varepsilon_{ut} & \varepsilon_{et} & \varepsilon_{rt} \end{bmatrix}'.$$

Following Binder and Pesaran (1985), decompose $s_{0,t}$ into "backward" and "forward" looking components:

$$s_{0,t} = s_{b,t} + s_{f,t} = D s_{0,t-1} + s_{f,t}, \tag{A.23}$$

where D is a 9×9 matrix to be determined. Substitute (A.23) into the left-hand side of (A.22) and

$$E_t s_{0,t+1} = D s_{0,t} + E_t s_{f,t+1} = D^2 s_{0,t-1} + D s_{f,t} + E_t s_{f,t+1}$$

into the right-hand side of (A.22) to obtain

$$[I_{(9\times 9)} - BD]s_{f,t} = (BD^2 - D + A)s_{0,t-1} + BE_t s_{f,t+1} + C\xi_t. \tag{A.24}$$

Equation (A.24) reveals that for $s_{f,t}$ to be purely forward looking, D must solve the second-order matrix polynomial equation

$$BD^2 - D + A = 0_{(9\times 9)}. \tag{A.25}$$

Once the solution for D is in hand, (A.24) can be written more simply as

$$s_{f,t} = FE_t s_{f,t+1} + G\xi_t, \tag{A.26}$$

where $F = [I_{(9\times 9)} - BD]^{-1}B$ and $G = [I_{(9\times 9)} - BD]^{-1}C$. Assuming all the eigenvalues of F are inside the unit circle and using (25), (A.26) can be solved forward to obtain

$$s_{f,t} = H\xi_t, \tag{A.27}$$

where the 9×5 matrix H is determined by

$$\text{vec}(H) = \left[I_{(45\times 45)} - (P' \otimes F)\right]^{-1} \text{vec}(G).$$

Substituting (A.27) into (A.23) now provides the solution

$$s_{0,t} = Ds_{0,t-1} + H\xi_t. \tag{26}$$

Finally, combining (25) and (26) yields

$$s_{t+1} = \Pi s_t + W\varepsilon_{t+1}, \tag{A.28}$$

where

$$s_t = \begin{bmatrix} s'_{0,t} & \xi'_t \end{bmatrix}' = \begin{bmatrix} \hat{y}_t & \hat{\pi}_t & \hat{r}_t & \hat{m}_t & \hat{g}_t & \hat{\mu}_t & \hat{q}_t & \hat{x}_t & \hat{\lambda}_t & \hat{a}_t & \hat{z}_t & \hat{u}_t & \hat{e}_t & \varepsilon_{rt} \end{bmatrix}',$$

$$\Pi = \begin{bmatrix} D & HP \\ 0_{(5\times 9)} & P \end{bmatrix},$$

and

$$W = \begin{bmatrix} H \\ I_{(5\times 5)} \end{bmatrix}.$$

It only remains to find the matrix D that solves (A.25). To accomplish this task, start by rewriting (A.22) as

$$KE_t s_{1,t+1} = Ls_{1,t} + M\xi_t, \tag{A.29}$$

where

$$s_{1,t} = \begin{bmatrix} s'_{0,t-1} & s'_{0,t} \end{bmatrix}',$$

$$K = \begin{bmatrix} I_{(9\times 9)} & 0_{(9\times 9)} \\ 0_{(9\times 9)} & B \end{bmatrix},$$

$$L = \begin{bmatrix} 0_{(9\times 9)} & I_{(9\times 9)} \\ -A & I_{(9\times 9)} \end{bmatrix},$$

and

$$M = \begin{bmatrix} 0_{(9\times 5)} \\ -C \end{bmatrix}.$$

Higham and Kim (2000) and Lan and Meyer-Godhe (2012) show that D can be found using the generalized Schur, or QZ, decomposition, which identifies unitary matrices Q and Z such that

$$QKZ = S$$

and

$$QLZ = T$$

are both upper triangular, where the generalized eigenvalues of L and K from (A.29) can be recovered as the ratios of the diagonal elements on T and S:

$$\lambda(L, K) = \{t_{ii}/s_{ii} \mid i = 1, 2, \ldots, 18\}.$$

The matrices Q, Z, S, and T can always be arranged so that the generalized eigenvalues appear in ascending order in absolute value. Note that there are nine predetermined, or lagged, values in the vector $s_{1,t}$. Thus, if nine of the generalized eigenvalues in $\lambda(L, K)$ lie inside the unit circle and nine of the generalized eigenvalues lie outside the unit circle, the system has a unique dynamically stable solution. If more than nine of the generalized eigenvalues in $\lambda(L, K)$ lie outside the unit circle, then the system has no stable solution; and if less than nine of the generalized eigenvalues lie outside the unit circle, then the system has multiple stable solutions. For details, see Blanchard and Kahn (1980) and Klein (2000).

Assuming that there are exactly nine generalized eigenvalues that lie outside the unit circle, partition the matrix Z into 9×9 blocks:

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}.$$

Then, according to Higham and Kim (2000) and Lan and Meyer-Godhe (2012),

$$D = Z_{21} Z_{11}^{-1} \tag{A.30}$$

will be the unique solution to (A.25) with all of its eigenvalues inside the unit circle, and the matrix F appearing in (A.26) will also have all of its eigenvalues inside the unit circle.

8.4 Imposing Zero Nominal Interest Rates

Kulish, Morley, and Robinson (2017) develop methods to solve and estimate the model over samples including the period from 2009:1 through 2015:4 when the Federal Reserve held short-term nominal interest rates in the US in a range near zero. Prior to and after the zero nominal interest rate period, the log-linearized model's solution is given by (A.28), as derived above. Let $t = T_1$ denote the start of the zero interest rate period, when the central bank replaces the Taylor rule (14) with the zero nominal interest rate condition (17). Then (17) can be combined with the remaining equilibrium conditions (A.9)-(A.13) and (A.15)-(A.21) to obtain

$$\bar{A}_0 s_{0,t} = \bar{J}_0 + \bar{A}_1 s_{0,t-1} + \bar{B}_0 E_t s_{0,t+1} + \bar{C}_0 \xi_t, \tag{27}$$

where the 9×9 matrices $\bar{A}_0$, $\bar{A}_1$, and $\bar{B}_0$ and the 9×5 matrix $\bar{C}_0$ coincide with A_0, A_1, B_0, and C_0 in (24), except that they replace with zeros the elements corresponding to the parameters of the Taylor rule (14), and the 9×1 vector $\bar{J}_0$ consists of zeros except for the term $-\ln(r)$ appearing in the row corresponding to the zero interest rate condition (17). After inverting $\bar{A}_0$, (27) can be written in the slightly simpler form

$$s_{0,t} = \bar{J} + \bar{A} s_{0,t-1} + \bar{B} E_t s_{0,t+1} + \bar{C} \xi_t, \tag{A.31}$$

where $\bar{J} = \bar{A}_0^{-1} \bar{J}_0$, $\bar{A} = \bar{A}_0^{-1} \bar{A}_1$, $\bar{B} = \bar{A}_0^{-1} \bar{B}_0$, and $\bar{C} = \bar{A}_0^{-1} \bar{C}_0$.

During periods $t = T_1, T_1 + 1, \ldots, T_2 = T_1 + \tau - 1$, while interest rates are held at zero, the solution for $s_{0,t}$ will have time-varying coefficients:

$$s_{0,t} = J_t + D_t s_{0,t-1} + H_t \xi_t, \tag{28}$$

which implies that

$$E_t s_{0,t+1} = J_{t+1} + D_{t+1} s_{0,t} + H_{t+1} P \xi_t. \tag{A.32}$$

Substitute (A.32) into (A.31) to obtain

$$s_{0,t} = \bar{J} + \bar{A} s_{0,t-1} + \bar{B} J_{t+1} + \bar{B} D_{t+1} s_{0,t} + \bar{B} H_{t+1} P \xi_t + \bar{C} \xi_t. \tag{A.33}$$

Matching coefficients across (28) and (A.33) then yields

$$D_t = [I_{(9\times9)} - \bar{B} D_{t+1}]^{-1} \bar{A}, \tag{A.34}$$

$$H_t = [I_{(9\times9)} - \bar{B} D_{t+1}]^{-1} (\bar{C} + \bar{B} H_{t+1} P), \tag{A.35}$$

and

$$J_t = [I_{(9\times9)} - \bar{B} D_{t+1}]^{-1} (\bar{J} + \bar{B} J_{t+1}). \tag{A.36}$$

Starting from the terminal conditions $D_{T_2+1} = D$ and $H_{T_2+1} = H$, where D is determined by (A.30) and H by (A.27), and $J_{T_2+1} = 0_{(9\times1)}$, (A.34)-(A.36) can be solved via backward recursion for the sequences $\{D_{T_1+j}\}_{j=0}^{\tau-1}$, $\{H_{T_1+j}\}_{j=0}^{\tau-1}$, and $\{J_{T_1+j}\}_{j=0}^{\tau-1}$, that appear in (28).

Still following Kulish, Morley, and Robinson (2017), assume more generally that the central bank re-evaluates the timing of its return to conventional policymaking via the Taylor rule (14) each period, announcing at the beginning of each time period t that the zero nominal interest rate episode will continue for τ_t more periods. To keep track of outcomes in this case, let $\bar{\tau}$ be an arbitrarily large upper bound on the length of the zero interest rate episode, and re-label the subscripts on the matrices that solve (A.34)-(A.36) to that $\{D_k\}_{k=1}^{\bar{\tau}}$, $\{H_k\}_{k=1}^{\bar{\tau}}$, and $\{J_k\}_{k=1}^{\bar{\tau}}$ are those that apply during any period when the zero interest rate episode is expected to last for k more periods. Now, the matrices that appear in the solution (28) for the zero interest rate episode are given by $D_t = D_{\tau_t}$, $H_t = H_{\tau_t}$, and $J_t = J_{\tau_t}$. And, as noted above, the model's solution before and after the zero interest rate episode can be written in the same form, where $D_t = D$, $H_t = H$, and $J_t = 0_{(9\times1)}$. Therefore, the solution for the model with the zero interest rate episode takes the form

$$s_{t+1} = \nu_{t+1} + \Pi_{t+1} s_t + W_{t+1}\varepsilon_{t+1}, \tag{A.37}$$

where

$$\nu_{t+1} = \begin{bmatrix} J_{t+1} \\ 0_{(5\times1)} \end{bmatrix},$$

$$\Pi_{t+1} = \begin{bmatrix} D_{t+1} & H_{t+1}P \\ 0_{(5\times9)} & P \end{bmatrix},$$

and

$$W_{t+1} = \begin{bmatrix} H_{t+1} \\ I_{(5\times5)} \end{bmatrix}.$$

8.5 The Likelihood Function

Let d_t denote the vector of observable variables at each date $t = 1, 2, \ldots, T$ in the sample period. Because the short-term nominal interest has zero variance according to the zero interest rate condition (17), it must be removed from d_t during the zero interest rate episode, from $t = T_1$ through $t = T_2$. Thus, in general,

$$d_t = U_t s_t, \tag{A.38}$$

where, outside the zero interest rate episode, U_t is 4×16 and formed from the selection vectors that pick out $\hat{g}_t$, $\hat{\pi}_t$, $\hat{r}_t$, and $\hat{\mu}_t$ from the state vector s_t and, during the zero interest rate episode, U_t is 3×16 and formed from the selection vectors that pick out just $\hat{g}_t$, $\hat{\pi}_t$, and $\hat{\mu}_t$ instead. The time-varying matrices entering into the state-space model formed by (A.37) and (A.38) can be accommodated within the standard Kalman filtering framework, as shown by Harvey (1989, Ch.3) and Anderson and Moore (2005, Ch.3), to evaluate the model's likelihood function $L(\{d_t\}_{t=1}^{T}|\Theta, \Delta)$.

In particular, let

$$\hat{s}_t = E(s_t|d_{t-1}, d_{t-2}, \ldots, d_1)$$

and

$$\Sigma_t = E(s_t - \hat{s}_t)(s_t - \hat{s}_t)'.$$

Then, starting from the initial conditions implied by (A.28),

$$\hat{s_1} = 0_{(14\times1)}$$

and

$$\text{vec}(\Sigma_1) = [I_{(196\times196)} - \Pi \otimes \Pi]^{-1}\text{vec}(WVW'),$$

generate recursively the sequences

$$\upsilon_t = d_t - U_t\hat{s}_t,$$

$$K_t = \Pi_{t+1}\Sigma_t U_t'(U_t\Sigma_t U_t')^{-1},$$

$$\hat{s}_{t+1} = \nu_{t+1} + \Pi_{t+1}\hat{s}_t + K_t \upsilon_t,$$

and

$$\Sigma_{t+1} = W_{t+1}VW_{t+1}' + \Pi_{t+1}\Sigma_t\Pi_{t+1}' - \Pi_{t+1}\Sigma_t U_t'(U_t\Sigma_t U_t')^{-1}U_t\Sigma_t\Pi_{t+1}',$$

where

$$V = E\varepsilon_{t+1}\varepsilon_{t+1}' = \begin{bmatrix} \sigma_a^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_z^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_u^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_e^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_r^2 \end{bmatrix}$$

is the covariance matrix of the New Keynesian model's structural shocks.

The innovations $\{\upsilon_t\}_{t=1}^T$ can then be used to evaluate likelihood function as

$$\begin{aligned} \ln(L(\{d_t\}_{t=1}^T|\Theta,\Delta) &= -\left[\frac{4(T-T_2+T_1-1)+3(T_2-T_1+1)}{2}\right]\ln(2\pi) \\ &\quad -\frac{1}{2}\sum_{t=1}^T \ln\left(|U_t\Sigma_t U_t'|\right) - \frac{1}{2}\sum_{t=1}^T \upsilon_t'(U_t\Sigma_t U_t')^{-1}\upsilon_t. \end{aligned}$$

8.6 Simulating the Posterior Distribution

The model's log posterior kernel can now be evaluated as

$$\ln L(\{d_t\}_{t=1}^T|\Theta,\Delta) + \ln(P(\Theta,\Delta)),$$

where $P(\Theta,\Delta)$ is the prior density over both sets of parameters. Kulish, Morley, and Robinson's (2017) modification of the randomized block Metropolis-Hastings algorithm of Chib and Ramamurthy (2010) is used to simulate draws from this posterior distribution.

The algorithm is initialized by finding the mode $\hat{\Theta}_0$ of the log posterior kernel, evaluated using data running from 1983:1 through 2008:4, that is, before the zero nominal interest rate episode, and Σ_Θ, minus one times the inverse of the matrix of second derivatives of the log posterior kernel, evaluated at this initial maximizer. Similarly, the mode of the prior distributions for each of the duration parameters is used to initialize $\hat{\Delta}_0$.

A random number n_Θ of the 17 parameters in Θ get updated in each iteration of the algorithm. First, n_Θ itself is chosen from a discrete uniform distribution over $[1, 17]$. Next, the specific n_Θ parameters to be updated are randomly chosen without replacement, again from a discrete uniform distribution over $[1, 17]$. Using $\Theta^{(1)}$ to denote the vector of parameters to be updated and $\Theta^{(2)}$ the vector of parameters that are not being updated, and given the previous draw $\hat{\Theta}_i = (\hat{\Theta}_i^{(1)}, \hat{\Theta}_i^{(2)})$, a new proposal $\Theta_{i+1}^{(1)}$ is drawn using a multivariate Student t distribution with location $\hat{\Theta}_i^{(1)}$, scale matrix based on a re-arrangement of Σ_Θ reflecting the partition $\Theta = (\Theta^{(1)}, \Theta^{(2)})$, ν degrees of freedom, and tuning parameter $\bar{\omega}$. In particular, Ding (2016) shows that if

$$\begin{bmatrix} \Theta_{i+1}^{(1)} \\ \Theta_{i+1}^{(2)} \end{bmatrix} \sim t\left(\begin{bmatrix} \Theta_{i}^{(1)} \\ \Theta_{i}^{(2)} \end{bmatrix}, \bar{\omega} \begin{bmatrix} \Sigma_{\Theta,11} & \Sigma_{\Theta,12} \\ \Sigma_{\Theta,21} & \Sigma_{\Theta,22} \end{bmatrix}, \nu \right),$$

then

$$\Theta_{i+1}^{(1)} | \Theta_{i+1}^{(2)} = \Theta_{i}^{(2)} \sim t\left(\Theta_{i}^{(1)}, \bar{\omega}\left(\frac{\nu}{\nu + p_2} \right) (\Sigma_{\Theta,11} - \Sigma_{\Theta,12}\Sigma_{\Theta,22}^{-1}\Sigma_{\Theta,21}), \nu + p_2 \right),$$

where $p_2 = 17 - n_\Theta$ is the number of elements not being updated.

With $\Theta_{i+1}^{(1)}$ drawn from this conditional distribution and with

$$\omega = \min \left\{ \frac{L(\{d_t\}_{t=1}^{T} | \Theta_{i+1}^{(1)}, \hat{\Theta}_{i}^{(2)}, \hat{\Delta}_i) P(\Theta_{i+1}^{(1)})}{L(\{d_t\}_{t=1}^{T} | \hat{\Theta}_{i}^{(1)}, \hat{\Theta}_{i}^{(2)}, \hat{\Delta}_i) P(\hat{\Theta}_{i}^{(1)})}, 1 \right\},$$

φ is drawn from a continuous uniform distribution on $(0, 1)$. If $\varphi > \omega$, the new draw is rejected by setting $\hat{\Theta}_{i+1} = \hat{\Theta}_i$. If $\varphi \leq \omega$, the new draw is accepted by setting $\hat{\Theta}_{i+1} = (\Theta_{i+1}^{(1)}, \hat{\Theta}_{i}^{(2)})$. Draws for the vector of structural parameters inconsistent with the existence of a unique dynamically stable solution to the New Keynesian model during periods outside of the zero nominal interest rate episode are automatically rejected, effectively truncating the prior distribution for these parameters at the boundary of the equilibrium determinacy region. In practice, $\nu = 12$ degrees of freedom is chosen for the Student t proposal distribution and the tuning parameter $\bar{\omega} = 0.6$ is set to achieve an acceptance rate of 0.27 for the draws of $\Theta_{i+1}^{(1)}$.

Likewise, a random number n_Δ of the 28 duration parameters in Δ get updated in each iteration. First, n_Δ is chosen from a discrete uniform distribution over $[1, l_\Delta]$, where $l_\Delta \leq 28$ is the maximum number of durations to be updated. Then, the specific n_Δ durations to be updated are randomly chosen without replacement from a uniform distribution over $[1, 28]$, since the zero nominal interest rate episode lasted 28 quarters from 2009:1 through 2015:4.

Let $\Delta^{(1)}$ denote the vector of durations to be updated and let $\Delta^{(2)}$ be the vector of durations that are not being updated. Given the previous draw $\hat{\Delta}_i = (\hat{\Delta}_i^{(1)}, \hat{\Delta}_i^{(2)})$, a new proposal $\Delta_{i+1}^{(1)}$ is drawn from a mixture $q(\Delta_{i+1}^{(1)})$ assigning weight 0.60 to the probabilities implied by survey data used to calibrate the prior and weight 0.40 to a uniform distribution over durations ranging from one through 23 quarters. In practice, this mixture is chosen, together with a setting of $l_\Delta = 6$, to achieve an acceptance rate of 0.24 for the draws of $\Delta_{i+1}^{(1)}$.

With $\Delta_{i+1}^{(1)}$ drawn from this distribution and with

$$\omega = \min \left\{ \frac{L(\{d_t\}_{t=1}^{T} | \hat{\Theta}_{i+1}, \Delta_{i+1}^{(1)}, \hat{\Delta}_{i}^{(2)}) P(\Delta_{i+1}^{(1)}) q(\hat{\Delta}_{i}^{(1)})}{L(\{d_t\}_{t=1}^{T} | \hat{\Theta}_{i+1}, \hat{\Delta}_{i}^{(1)}, \hat{\Delta}_{i}^{(2)}) P(\hat{\Delta}_{i}^{(1)}) q(\Delta_{i+1}^{(1)})}, 1 \right\},$$

φ is drawn from a continuous uniform distribution on $(0, 1)$. If $\varphi > \omega$, the new draw is rejected by setting $\hat{\Delta}_{i+1} = \hat{\Delta}_i$. If $\varphi \leq \omega$, the new draw is accepted by setting $\hat{\Delta}_{i+1} = (\Delta_{i+1}^{(1)}, \hat{\Delta}_{i}^{(2)})$.

All results in the paper are based on 1 million draws taken from a simulation consisting of 11 million iterations of this algorithm. The first 1 million draws are discarded to allow for burn-in, and one out of every ten of the remaining 10 million draws are stored to generate the final results.

8.7 The Marginal Likelihood

The model's marginal likelihood is the expected value of the likelihood with respect to the prior distribution:

$$P(\{d_t\}_{t=1}^T) = \int\int L(\{d_t\}_{t=1}^T|\Theta, \Delta)P(\Theta, \Delta)\, \mathrm{d}\Theta\mathrm{d}\Delta.$$

It can be evaluated using the modified harmonic mean approach suggested by Gelfand and Dey (1994) and Geweke (1999). Here, however, the approach must be modified slightly, since the elements of Δ are restricted to the positive integers.

First, for the continuously-valued parameters in Θ, follow Geweke (1999, pp.46-7), by letting $\bar{\Theta}$ and Ω_Θ denote the mean and covariance matrix computed from the posterior draws $\{\hat{\Theta}_i\}_{i=1}^N$, where N denotes the total number of saved draws. Let g_θ denote the pdf for a truncated multivariate normal, where the truncation points are determined by the pth percentile of a chi-squared random variable with k degrees of freedom, where k is the length of the parameter vector Θ. That is, let

$$g_\theta(\Theta) = \frac{1}{p}\left(\frac{1}{2\pi}\right)^{k/2} |\Omega_\Theta|^{-1/2} \exp[-(1/2)(\Theta - \bar{\Theta})'\Omega_\Theta^{-1}(\Theta - \bar{\Theta})]$$

if

$$(\Theta - \bar{\Theta})'\Omega_\Theta^{-1}(\Theta - \bar{\Theta}) \leq F_{\chi_k^2}^{-1}(p),$$

where $F_{\chi_k^2}^{-1}(\cdot)$ is the inverse cdf for a chi-squared random variable with k degrees of freedom, and $g_\theta(\Theta) = 0$ otherwise.

Next, for the discretely-value parameters in Δ, let $g_\delta(\cdot)$ coincide with the probability mass function implied by the survey data on durations of the zero lower bound episode. Since the prior distribution of the elements of Δ is a combination of this distribution and a uniform distribution over all durations from 1 to 23 quarters, this choice accomplishes the same goal as Geweke's truncated normal for the elements of Θ, by providing heavier overlap with and avoiding the tails of the posterior.

Now the marginal likelihood can be approximated from the output of the Markov chain Monte Carlo sampling algorithm using

$$P(\{d_t\}_{t=1}^T) \approx \left[\frac{1}{N}\sum_{i=1}^N \frac{g_\theta(\hat{\Theta}_i)g_\delta(\hat{\Delta}_i)}{L(\{d_t\}_{t=1}^T|\hat{\Theta}_i, \hat{\Delta}_i)P(\hat{\Theta}_i, \hat{\Delta}_i)}\right]^{-1}.$$

All of the results reported in table 3 are robust to choices of p ranging from 0.9 to 0.999.

8.8 Draws for the Unobservable States and Innovations

The algorithm described above can be extended to obtain draws for the unobservable states and innovations using the simulation-smoothing method developed by Durbin and Koopman (2002) and outlined in Durbin and Koopman (2012, pp.107-8).

From the state-space formulation derived earlier as

$$s_{t+1} = \nu_{t+1} + \Pi_{t+1} s_t + W_{t+1}\varepsilon_{t+1} \tag{A.37}$$

and

$$d_t = U_t s_t, \tag{A.38}$$

Kalman filter recursions starting from

$$\hat{s_1} = 0_{(14\times 1)}$$

and

$$\text{vec}(\Sigma_1) = [I_{(196\times 196)} - \Pi \otimes \Pi]^{-1}\text{vec}(WVW'),$$

can be used to generate recursively the sequences

$$\upsilon_t = d_t - U_t \hat{s}_t,$$

$$\Omega_t = U_t \Sigma_t U_t',$$

$$K_t = \Pi_{t+1}\Sigma_t U_t' \Omega_t^{-1},$$

$$L_t = \Pi_{t+1} - K_t U_t,$$

$$\hat{s}_{t+1} = \nu_{t+1} + \Pi_{t+1}\hat{s}_t + K_t \upsilon_t,$$

and

$$\Sigma_{t+1} = W_{t+1} V W_{t+1}' + \Pi_{t+1}\Sigma_t L_t'$$

for $t = 1, 2, \ldots, T$. From this filtering stage, the sequences for υ_t, Ω_t, and K_t are stored.

Next, a value s_1^a is drawn from the distribution $N(0_{(14\times 1)}, \Sigma_1)$ for the initial state and a series of artificial shocks $\{\varepsilon_t\}_{t=1}^T$ drawn from $N(0_{(5\times 1)}, V)$ for all $t = 1, 2, \ldots, T$. When fed through (A.37) and (A.38), these draws create an artificial data series $\{d_t^a\}_{t=1}^T$ that can also be run through the Kalman filter to generate sequences υ_t^a, Ω_t^a, and K_t^a.

For both the actual and artificial series, the terminal value $n_T = 0_{(14\times 1)}$ gets used to perform the backwards recursion

$$n_{t-1} = U_t' \Omega_t^{-1} \upsilon_t + L_t' n_t,$$

where L_t is again given by

$$L_t = \Pi_{t+1} - K_t U_t,$$

yielding the sequences $\{n_t\}_{t=0}^T$ for the actual and $\{n_t^a\}_{t=0}^T$ for the artificial data series. These sequences are, in turn, used to generate $\{\tilde{\varepsilon}_t\}_{t=1}^{T+1}$ using

$$\tilde{\varepsilon}_{t+1} = V W_{t+1}' n_t$$

and the series $\{\tilde{\varepsilon}^a_t\}_{t=1}^{T+1}$ using

$$\tilde{\varepsilon}^a_{t+1} = VW'_{t+1}n^a_t.$$

Durbin and Koopman show that the sequence $\{\hat{\varepsilon}_t\}_{t=1}^{T+1}$ constructed using

$$\hat{\varepsilon}_{t+1} = \varepsilon^a_{t+1} - \tilde{\varepsilon}^a_{t+1} + \tilde{\varepsilon}_{t+1}$$

are draws from the posterior distribution of the vector $\{\varepsilon_t\}_{t=1}^{T+1}$ of innovations to the New Keynesian model's structural shocks, conditional on the entire series of observed data $\{d_t\}_{t=1}^{T}$.

Similarly, draws from the posterior distribution of the unobservable state vector $\{s_t\}_{t=1}^{T+1}$ can be taken by first drawing s^a_1 from the distribution $N(0_{(14\times 1)}, \Sigma_1)$ and $\{\varepsilon^a_t\}_{t=1}^{T}$ from $N(0, V)$ for all $t = 0, 1, \ldots, T$, and using these to generate a series of artificial states $\{s^a_t\}_{t=1}^{T+1}$ with (A.37). Then, with $\{n_t\}_{t=0}^{T}$ and $\{n^a_t\}_{t=0}^{T}$ obtained from the backward recursions above, the sequence $\{\tilde{s}_t\}_{t=1}^{T+1}$ is generated from

$$\tilde{s}_{t+1} = \nu_{t+1} + \Pi_{t+1}\tilde{s}_t + W_{t+1}\tilde{\varepsilon}_{t+1}$$

starting from $\tilde{s}_1 = \Sigma_1 n_0$ and the sequence $\{\tilde{s}^a_t\}_{t=1}^{T+1}$ is generated from

$$\tilde{s}^a_{t+1} = \nu_{t+1} + \Pi_{t+1}\tilde{s}^a_t + W_{t+1}\tilde{\varepsilon}^a_{t+1}$$

starting from $\tilde{s}^a_1 = \Sigma_1 n^a_0$. Durbin and Koopman show that the sequence $\{\hat{s}_t\}_{t=1}^{T+1}$ constructed using

$$\hat{s}_{t+1} = s^a_{t+1} - \tilde{s}^a_{t+1} + \tilde{s}_{t+1}$$

are draws from the distribution of $\{s_t\}_{t=1}^{T+1}$ conditional on $\{d_t\}_{t=1}^{T}$.

8.9 Additional References

Binder, Michael and M. Hashem Pesaran. "Multivariate Rational Expectations Models and Macroeconometric Modelling: A Review and Some New Results." In M. Hashem Pesaran and Mike Wickens, Eds. *Handbook of Applied Econometrics: Macroeconomics.* Oxford: Basil Blackwell, 1995, 139-187.

Blanchard, Olivier Jean and Charles M, Kahn. "The Solution of Linear Difference Models Under Rational Expectations." *Econometrica* 48 (July 1980): 1305-1311.

Ding, Peng. "On The Conditional Distribution of the Multivariate t Distribution." *The American Statistician* 70 (Issue 3, 2016): 293-295.

Durbin, J. and S.J. Koopman. *Time Series Analysis by State Space Methods* 2d Ed. Oxford: Oxford University Press, 2012.

Gelfand, A.E. and D.K. Dey. "Bayesian Model Choice: Asymptotics and Exact Calculations." *Journal of the Royal Statistical Society, Series B (Methodological)* 56 (1994, Number 3): 501-514.

Higham, Nicholas J. and Hyun-Min Kim. "Numerical Analysis of a Quadratic Matrix Equation." *IMA Journal of Numerical Analysis* 20 (October 2000): 499-519.

Klein, Paul. "Using the Generalized Schur Form to Solve a Multivariate Linear Rational Expectations Model." *Journal of Economic Dynamics and Control* 24 (September 2000): 1405-1423.

Lan, Hong and Alexander Meyer-Gohde. "Existence and Uniqueness of Perturbation Solutions to DSGE Models." SFB 649 Discussion Paper 2012-015. Berlin: Humboldt University, February 2012.