Value Function Iteration

Consider the discrete-time Ramsey model with log utility, Cobb-Douglas production, and complete depreciation $\delta = 1$.

Utility:

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

Capital accumulation:

$$k_{t+1} = k_t + k_t^{\alpha} - \delta k_t - c_t = k_t^{\alpha} - c_t$$

or allowing for free disposal

$$k_t^{\alpha} \geq c_t + k_{t+1}$$

Value Function Iteration

The social planner's problem: given k_0, choose $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$k_t^{\alpha} \geq c_t + k_{t+1}$$

for all $t = 0, 1, 2, \ldots$.

Value Function Iteration

Given k_0, choose $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$k_t^{\alpha} \geq c_t + k_{t+1}$$

for all $t = 0, 1, 2, \ldots$.

The Bellman equation:

$$v(k_t; t) = \max_{c_t} \ln(c_t) + \beta v(k_t^{\alpha} - c_t; t + 1)$$

Value Function Iteration

The Bellman equation:

$$v(k_t; t) = \max_{c_t} \ln(c_t) + \beta v(k_t^\alpha - c_t; t + 1)$$

For the Ramsey model in general:

$$v(k_t; t) = v(k_t)$$

And with log utility, Cobb-Douglas production, and $\delta = 1$:

$$v(k_t) = E + F \ln(k_t)$$

where E and F are "undetermined coefficients."

Value Function Iteration

$$v(k_t; t) = \max_{c_t} \ln(c_t) + \beta v(k_t^\alpha - c_t; t+1)$$

And with log utility, Cobb-Douglas production, and $\delta = 1$:

$$v(k_t; t) = v(k_t) = E + F \ln(k_t)$$

where guess-and-verify reveals that

$$E = \frac{1}{1-\beta} \left[\ln(1 - \alpha\beta) + \frac{\alpha\beta}{1 - \alpha\beta} \ln(\alpha\beta) \right]$$

and

$$F = \frac{\alpha}{1 - \alpha\beta}$$

Value Function Iteration

Where, besides some combination of skill and luck, might the conjectured form of the value function come from?

Consider taking an iterative approach. Let $v_0(k_t)$ be an initial guess, then compute

$$v_1(k_t) = \max_{c_t} \ln(c_t) + \beta v_0(k_t^\alpha - c_t)$$

Value Function Iteration

Suppose, in particular, that $v_0(k_t) = 0$. Then

$$v_1(k_t) = \max_{c_t} \ln(c_t) + \beta v_0(k_t^\alpha - c_t)$$

becomes

$$\begin{aligned} v_1(k_t) &= \max_{c_t} \ln(c_t) \text{ subject to } k_t^\alpha \geq c_t \\ &= \ln(k_t^\alpha) \\ &= \alpha \ln(k_t) \end{aligned}$$

Value Function Iteration

Now use the updated guess $v_1(k_t) = \alpha \ln(k_t)$ to find

$$\begin{aligned} v_2(k_t) &= \max_{c_t} \ln(c_t) + \beta v_1(k_t^\alpha - c_t) \\ &= \max_{c_t} \ln(c_t) + \alpha\beta \ln(k_t^\alpha - c_t) \end{aligned}$$

FOC for c_t:

$$\frac{1}{c_t} - \frac{\alpha\beta}{k_t^\alpha - c_t} = 0$$

implies

$$c_t = \left(\frac{1}{1+\alpha\beta}\right) k_t^\alpha$$

Value Function Iteration

$$
\begin{aligned}
v_2(k_t) &= \max_{c_t} \ln(c_t) + \beta v_1(k_t^\alpha - c_t) \\
&= \max_{c_t} \ln(c_t) + \alpha\beta \ln(k_t^\alpha - c_t) \\
&= \ln\left[\left(\frac{1}{1+\alpha\beta}\right) k_t^\alpha\right] + \alpha\beta \ln\left[\left(\frac{\alpha\beta}{1+\alpha\beta}\right) k_t^\alpha\right] \\
&= E_2 + (1+\alpha\beta)\alpha \ln(k_t)
\end{aligned}
$$

where

$$
E_2 = \ln\left(\frac{1}{1+\alpha\beta}\right) + \alpha\beta \ln\left(\frac{\alpha\beta}{1+\alpha\beta}\right)
$$

Value Function Iteration

Now use the updated guess $v_2(k_t) = E_2 + (1 + \alpha\beta)\alpha \ln(k_t)$ to find

$$\begin{aligned} v_3(k_t) &= \max_{c_t} \ln(c_t) + \beta v_2(k_t^\alpha - c_t) \\ &= \max_{c_t} \ln(c_t) + \beta E_2 + (1 + \alpha\beta)\alpha\beta \ln(k_t^\alpha - c_t) \end{aligned}$$

FOC for c_t:

$$\frac{1}{c_t} - \frac{(1 + \alpha\beta)\alpha\beta}{k_t^\alpha - c_t} = 0$$

implies

$$c_t = \left[\frac{1}{1 + \alpha\beta + (\alpha\beta)^2}\right] k_t^\alpha$$

Value Function Iteration

$$
\begin{aligned}
v_3(k_t) &= \max_{c_t} \ln(c_t) + \beta v_2(k_t^\alpha - c_t) \\
&= \max_{c_t} \ln(c_t) + \beta E_2 + (1+\alpha\beta)\alpha\beta \ln(k_t^\alpha - c_t) \\
&= \ln\left\{\left[\frac{1}{1+\alpha\beta+(\alpha\beta)^2}\right] k_t^\alpha\right\} + \beta E_2 \\
&\quad + (1+\alpha\beta)\alpha\beta \ln\left\{\left[\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}\right] k_t^\alpha\right\} \\
&= E_3 + \alpha[1+\alpha\beta+(\alpha\beta)^2]\ln(k_t)
\end{aligned}
$$

where

$$
E_3 = \beta E_2 + \ln\left[\frac{1}{1+\alpha\beta+(\alpha\beta)^2}\right] + (1+\alpha\beta)\alpha\beta \ln\left[\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}\right]
$$

Value Function Iteration

Use the updated guess $v_3(k_t) = E_3 + [1 + \alpha\beta + (\alpha\beta)^2]\alpha \ln(k_t)$ to find

$$\begin{aligned} v_4(k_t) &= \max_{c_t} \ln(c_t) + \beta v_3(k_t^\alpha - c_t) \\ &= \max_{c_t} \ln(c_t) + \beta E_3 + [1 + \alpha\beta + (\alpha\beta)^2]\alpha\beta \ln(k_t^\alpha - c_t) \end{aligned}$$

FOC for c_t:

$$\frac{1}{c_t} - \frac{[1 + \alpha\beta + (\alpha\beta)^2]\alpha\beta}{k_t^\alpha - c_t} = 0$$

implies

$$c_t = \left[\frac{1}{1 + \alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3}\right] k_t^\alpha$$

Value Function Iteration

$$\begin{aligned}
v_4(k_t) &= \max_{c_t} \ln(c_t) + \beta v_3(k_t^\alpha - c_t) \\
&= \max_{c_t} \ln(c_t) + \beta E_3 + [1 + \alpha\beta + (\alpha\beta)^2]\alpha\beta \ln(k_t^\alpha - c_t) \\
&= \ln\left\{\left[\frac{1}{1 + \alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3}\right] k_t^\alpha\right\} + \beta E_3 \\
&\quad + [1 + \alpha\beta + (\alpha\beta)^2]\alpha\beta \ln\left\{\left[\frac{\alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3}{1 + \alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3}\right] k_t^\alpha\right\} \\
&= E_4 + [1 + \alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3]\alpha \ln(k_t)
\end{aligned}$$

Value Function Iteration

$$
\begin{aligned}
v_4(k_t) &= \max_{c_t} \ln(c_t) + \beta v_3(k_t^\alpha - c_t) \\
&= E_4 + [1 + \alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3]\alpha \ln(k_t)
\end{aligned}
$$

A pattern in the coefficient on $\ln(k_t)$ seems to be emerging that will imply

$$
v_n(k_t) = E_n + \alpha \left[\sum_{j=0}^{n-1} (\alpha\beta)^j \right] \ln(k_t)
$$

Value Function Iteration

A pattern seems to be emerging that will imply

$$v_n(k_t) = E_n + \alpha \left[\sum_{j=0}^{n-1} (\alpha\beta)^j \right] \ln(k_t)$$

and therefore

$$\begin{aligned} \lim_{n\to\infty} v_n(k_t) &= \lim_{n\to\infty} E_n + \alpha \left[\sum_{j=0}^{\infty} (\alpha\beta)^j \right] \ln(k_t) \\ &= \lim_{n\to\infty} E_n + \left(\frac{\alpha}{1-\alpha\beta} \right) \ln(k_t) \\ &= E + F \ln(k_t) \end{aligned}$$

Value Function Iteration

The value function $v(k_t)$ is a fixed point of the functional equation

$$v(k_t) = \max_{c_t} \ln(c_t) + \beta v(k_t^\alpha - c_t)$$

that can be found either through guess-and-verify or through iteration on

$$v_{n+1}(k_t) = \max_{c_t} \ln(c_t) + \beta v_n(k_t^\alpha - c_t).$$

Value function iteration can be used to solve the Bellman equation numerically on a computer, in cases where no closed-form expression for the value function exists.