Natural Resource Depletion

Given s_0, choose $\{c_t\}_{t=0}^{T}$ and $\{s_t\}_{t=1}^{T+1}$ to maximize

$$\sum_{t=0}^{T} \beta^t \ln(c_t)$$

subject to

$$s_t - c_t \geq s_{t+1}$$

for all $t = 0, 1, \ldots, T$ and

$$s_{T+1} \geq 0$$

Natural Resource Depletion

$$L(\{c_t\}_{t=0}^{T}, \{s_t\}_{t=1}^{T+1}, \{\lambda_t\}_{t=1}^{T+1}, \phi)$$
$$= \sum_{t=0}^{T} \beta^t \ln(c_t) + \sum_{t=0}^{T} \beta^t \lambda_{t+1}(s_t - c_t - s_{t+1}) + \phi s_{T+1}$$

Note that the Lagrange multiplier λ_t is in "current value" form.

Natural Resource Depletion

$$L(\{c_t\}_{t=0}^{T}, \{s_t\}_{t=1}^{T+1}, \{\lambda_t\}_{t=1}^{T+1}, \phi)$$
$$= \sum_{t=0}^{T} \beta^t \ln(c_t) + \sum_{t=0}^{T} \beta^t \lambda_{t+1}(s_t - c_t - s_{t+1}) + \phi s_{T+1}$$

FOC for c_t:

$$\frac{\beta^t}{c_t} - \beta^t \lambda_{t+1} = 0$$

or

$$\frac{1}{c_t} = \lambda_{t+1} \tag{1}$$

for all $t = 0, 1, \ldots, T$

Natural Resource Depletion

$$L(\{c_t\}_{t=0}^{T}, \{s_t\}_{t=1}^{T+1}, \{\lambda_t\}_{t=1}^{T+1}, \phi)$$
$$= \sum_{t=0}^{T} \beta^t \ln(c_t) + \sum_{t=0}^{T} \beta^t \lambda_{t+1}(s_t - c_t - s_{t+1}) + \phi s_{T+1}$$

FOC for s_t:

$$\beta^t \lambda_{t+1} - \beta^{t-1} \lambda_t = 0$$

or

$$\beta \lambda_{t+1} = \lambda_t \quad (2)$$

for all $t = 1, 2, \ldots, T$

Natural Resource Depletion

$$L(\{c_t\}_{t=0}^{T}, \{s_t\}_{t=1}^{T+1}, \{\lambda_t\}_{t=1}^{T+1}, \phi)$$
$$= \sum_{t=0}^{T} \beta^t \ln(c_t) + \sum_{t=0}^{T} \beta^t \lambda_{t+1}(s_t - c_t - s_{t+1}) + \phi s_{T+1}$$

FOC for s_{T+1}:

$$-\beta^T \lambda_{T+1} + \phi = 0$$

or

$$\phi = \beta^T \lambda_{T+1}.$$

Natural Resource Depletion

The FOCs

$$\frac{1}{c_t} = \lambda_{t+1} \tag{1}$$

$$\beta\lambda_{t+1} = \lambda_t \tag{2}$$

and the binding constraint

$$s_{t+1} = s_t - c_t \tag{3}$$

form a system of three equations in three unknowns: c_t, s_t, and λ_t.

Natural Resource Depletion

The pair of difference equations (2) and (3) must be solved subject to two boundary conditions.
The initial condition

$$s_0 \text{ given}$$

And the TVC

$$\phi s_{T+1} = \beta^T \lambda_{T+1} s_{T+1} = 0.$$

With a finite horizon, the TVC is just the complementary slackness condition associated with the constraint $s_{T+1} > 0$ on the terminal stock.

Natural Resource Depletion

Given s_0, choose $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$s_t - c_t \geq s_{t+1}$$

for all $t = 0, 1, 2, \ldots$.

With an infinite horizon, there is no "terminal" stock! But there is still a TVC.

Natural Resource Depletion

The FOCs

$$\frac{1}{c_t} = \lambda_{t+1} \tag{1}$$

for all $t = 0, 1, 2, \ldots$ and

$$\beta\lambda_{t+1} = \lambda_t \tag{2}$$

for all $t = 1, 2, 3, \ldots$, and the binding constraint

$$s_{t+1} = s_t - c_t \tag{3}$$

or all $t = 0, 1, 2, \ldots$, still form a system of three equations in three unknowns: c_t, s_t, and λ_t.

Natural Resource Depletion

The pair of difference equations (2) and (3) must still be solved subject to two boundary conditions: The initial condition

$$s_0 \text{ given}$$

and the TVC

$$\lim_{T \to \infty} \beta^T \lambda_{T+1} s_{T+1} = 0 \tag{4}$$

The TVC for the infinite-horizon problem is the limit as $T \to \infty$ of the TVC in the finite-horizon model.

Natural Resource Depletion

In this case, showing that the TVC at infinity

$$\lim_{T\to\infty} \beta^T \lambda_{T+1} s_{T+1} = 0 \tag{4}$$

is a necessary condition for optimality requires only a two-step proof.

Natural Resource Depletion

Note first that the sequence

$$\{\beta^t \lambda_{t+1} s_{t+1}\}_{t=0}^{\infty}$$

must be nonincreasing, since

$$\begin{aligned}
\beta^t \lambda_{t+1} s_{t+1} - \beta^{t-1} \lambda_t s_t &= \beta^t \lambda_{t+1} s_{t+1} - \beta^{t-1} \beta \lambda_{t+1} s_t \text{ by (2)} \\
&= \beta^t \lambda_{t+1} (s_{t+1} - s_t) \\
&\leq \beta^t \lambda_{t+1} (s_t - c_t - s_t) \text{ by the constraint} \\
&= -\beta^t \lambda_{t+1} c_t \\
&\leq 0
\end{aligned}$$

Natural Resource Depletion

Note second that

$$\inf_t \beta^t \lambda_{t+1} s_{t+1} = 0$$

To see this, suppose to the contrary that there exists $\varepsilon > 0$ such that

$$\beta^t \lambda_{t+1} s_{t+1} \geq \varepsilon$$

for all $t = 0, 1, 2, \ldots$

Natural Resource Depletion

Since

$$\beta\lambda_{t+1} = \lambda_t \tag{2}$$

implies

$$\beta^t\lambda_{t+1} = \beta^{t-1}\lambda_t$$

$\{\beta^t\lambda_{t+1}\}_{t=0}^{\infty}$ must be a constant sequence. Therefore,

$$\beta^t\lambda_{t+1}s_{t+1} \geq \varepsilon$$

requires that there exists a $\gamma > 0$ such that

$$s_{t+1} \geq \gamma$$

for all $t = 0, 1, 2, \ldots$.

Natural Resource Depletion

But if

$$s_{t+1} \geq \gamma$$

for all $t = 0, 1, 2, \ldots$, then we can define new sequences with

$$\tilde{c}_0 = c_0 + \gamma$$

$$\tilde{c}_t = c_t \text{ for all } t = 1, 2, 3, \ldots$$

$$\tilde{s}_t = s_t - \gamma \text{ for all } t = 1, 2, 3, \ldots$$

that satisfy all the constraints but yield higher utility.

Natural Resource Depletion

Together,

$$\{\beta^t \lambda_{t+1} s_{t+1}\}_{t=0}^{\infty}$$

nonincreasing and

$$\inf_t \beta^t \lambda_{t+1} s_{t+1} = 0$$

imply that the TVC

$$\lim_{T \to \infty} \beta^T \lambda_{T+1} s_{T+1} = 0 \tag{4}$$

must hold at the optimum.

Natural Resource Depletion

Notice for this problem that

$$\beta^T \lambda_{T+1} s_{T+1} = 0$$

holds for the finite horizon because $s_{T+1} = 0$ and

$$\lim_{T \to \infty} \beta^T \lambda_{T+1} s_{T+1} = 0 \qquad (4)$$

holds for the infinite horizon problem because

$$\lim_{T \to \infty} s_{T+1} = 0$$

Natural Resource Depletion

For the Ramsey model,

$$\beta^T \lambda_{T+1} k_{T+1} = 0$$

holds for the finite horizon because $k_{T+1} = 0$ but

$$\lim_{T \to \infty} \beta^T \lambda_{T+1} k_{T+1} = 0$$

holds for the infinite horizon problem because

$$\lim_{T \to \infty} \beta^T \lambda_{T+1} = 0 \text{ while } \lim_{T \to \infty} k_{T+1} = k^* > 0$$

Proving the necessity of the TVC at infinity is much harder!