NPG and TVC

Consider a bank account balance $B(t)$ that pays interest at the time-varying rate $r(t)$.

If no deposits or withdrawals are made after $t = 0$,

$$\dot{B}(t) = r(t)B(t)$$

NPG and TVC

Now define

$$R(t) = \exp\left[-\int_0^t r(s)ds\right]$$

By the chain rule and then Leibniz's rule,

$$\begin{aligned}\dot{R}(t) = & \exp\left[-\int_0^t r(s)ds\right]\left\{\frac{d}{dt}\left[-\int_0^t r(s)ds\right]\right\} \\ = & -r(t)R(t)\end{aligned}$$

NPG and TVC

Consider next this integration by parts argument:

$$\frac{dR(t)B(t)}{dt} = \dot{R}(t)B(t) + R(t)\dot{B}(t)$$

$$\int_0^T \left[\frac{dR(t)B(t)}{dt}\right] dt = \int_0^T \dot{R}(t)B(t)dt + \int_0^T R(t)\dot{B}(t)dt$$

$$R(T)B(T) - R(0)B(0) = \int_0^T \dot{R}(t)B(t)dt + \int_0^T R(t)\dot{B}(t)dt$$

NPG and TVC

Since

$$R(0) = \exp\left[-\int_0^0 r(s)ds\right] = \exp(0) = 1$$

and $\dot{R}(t) = -r(t)R(t)$,

$$R(T)B(T) - R(0)B(0) = \int_0^T \dot{R}(t)B(t)dt + \int_0^T R(t)\dot{B}(t)dt$$

implies

$$\int_0^T R(t)\dot{B}(t)dt = R(T)B(T) - B(0) + \int_0^T r(t)R(t)B(t)dt$$

NPG and TVC

Return to the bank account balance

$$\dot{B}(t) = r(t)B(t)$$

$$R(t)\dot{B}(t) = r(t)R(t)B(t)$$

$$\int_0^T R(t)\dot{B}(t)dt = \int_0^T r(t)R(t)B(t)dt$$

NPG and TVC

$$\int_0^T R(t)\dot{B}(t)dt = \int_0^T r(t)R(t)B(t)dt$$

and

$$\int_0^T R(t)\dot{B}(t)dt = R(T)B(T) - B(0) + \int_0^T r(t)R(t)B(t)dt$$

imply

$$R(T)B(T) = B(0)$$

NPG and TVC

$$R(T)B(T) = B(0)$$

$B(0)$ deposited at $t = 0$ is worth $R(T)B(T)$ at $t = T$

A bank account balance worth $B(T)$ at $t = T$ is worth $B(0)$ at $t = 0$.

$R(T)$ is a discount factor that converts dollars received at $t = T$ into present values at $t = 0$.

NPG and TVC

$$R(T)B(T) = B(0)$$

Converting to discrete time and assuming a constant interest rate

$$R(T) = \frac{1}{(1+r)^T}$$

NPG and TVC

Now elaborate, by letting a consumer make additional deposits or withdrawals based on the difference between his or her income $W(t)$ and consumption $C(t)$:

$$\dot{B}(t) = r(t)B(t) + W(t) - C(t)$$

Rearraging

$$r(t)B(t) + W(t) = C(t) + \dot{B}(t)$$

NPG and TVC

Let's convert the single-period budget constraint into a "lifetime" present value budget constraint:

$$r(t)B(t) + W(t) = C(t) + \dot{B}(t)$$

$$r(t)R(t)B(t) + R(t)W(t) = R(t)C(t) + R(t)\dot{B}(t)$$

$$\int_0^T r(t)R(t)B(t)dt + \int_0^T R(t)W(t)dt$$
$$= \int_0^T R(t)C(t)dt + \int_0^T R(t)\dot{B}(t)dt$$

NPG and TVC

But remember from integration by parts that

$$\int_0^T R(t)\dot{B}(t)dt = R(T)B(T) - B(0) + \int_0^T r(t)R(t)B(t)dt$$

Therefore

$$\begin{aligned} &\int_0^T r(t)R(t)B(t)dt + \int_0^T R(t)W(t)dt \\ = &\int_0^T R(t)C(t)dt + \int_0^T R(t)\dot{B}(t)dt \end{aligned}$$

simplifies to

$$B(0) + \int_0^T R(t)W(t)dt = \int_0^T R(t)C(t)dt + R(T)B(T)$$

NPG and TVC

$$B(0) + \int_0^T R(t)W(t)dt = \int_0^T R(t)C(t)dt + R(T)B(T)$$

We can allow the consumer to borrow, that is, we can allow $B(t) < 0$ for $t \in [0, T)$. But the no-Ponzi-game constraint requires

$$R(T)B(T) \geq 0.$$

Whatever gets borrowed must be paid back.

NPG and TVC

Therefore, the single-period budget constraint

$$r(t)B(t) + W(t) = C(t) + \dot{B}(t)$$

and the NPG constraint

$$R(T)B(T) \geq 0$$

imply that the consumer faces a lifetime present value budget constraint

$$B(0) + \int_0^T R(t)W(t)dt \geq \int_0^T R(t)C(t)dt$$

NPG and TVC

$$B(0) + \int_0^T R(t)W(t)dt = \int_0^T R(t)C(t)dt + R(T)B(T)$$

NPG requires $R(T)B(T) \geq 0$. But consumer optimality requires

$$R(T)B(T) \leq 0$$

The proof is by contradiction: if $R(T)B(T) > 0$, then $B(T) > 0$. Dying with money in the bank means you haven't consumed enough!

NPG and TVC

NPG requires $R(T)B(T) \geq 0$. But consumer optimality requires

$$R(T)B(T) \leq 0$$

Together, these two conditions imply that the transversality condition

$$R(T)B(T) = 0$$

must hold.

NPG and TVC

$$B(0) + \int_0^T R(t)W(t)dt = \int_0^T R(t)C(t)dt + R(T)B(T)$$

Whereas the NPG

$$R(T)B(T) \geq 0$$

implies the consumer faces an lifetime present value budget constraint, the TVC But consumer optimality requires

$$R(T)B(T) = 0$$

implies that the lifetime present value budget constraint binds at the optimum:

$$B(0) + \int_0^T R(t)W(t)dt = \int_0^T R(t)C(t)dt$$

NPG and TVC

Therefore, the NPG constraint

$$R(T)B(T) \geq 0$$

is a constraint that is imposed on the consumer: whatever you borrow, you have to pay back.

Whereas the TVC

$$R(T)B(T) = 0$$

is a necessary condition for consumer optimality: if it doesn't hold, the consumer is overaccumulating assets.

NPG and TVC

With a finite horizon, the NPG constraint is often relatively easy to formulate and the necessity of the TVC is relatively easy to establish.

With an infinite horizon, things become tricker. What exactly does it mean to say that “all borrowing must eventually be repaid” when the infinite horizon never ends?

NPG and TVC

Note, however, that so long as there is some finite limit on borrowing, with

$$B(t) \geq -\bar{B}$$

for some $\bar{B} > 0$, then

$$\lim_{T\to\infty} R(T)B(T) = \lim_{T\to\infty} \exp\left[-\int_0^T r(s)ds\right] B(T) \geq 0$$

will have to hold.

NPG and TVC

$$B(0) + \int_0^T R(t)W(t)dt = \int_0^T R(t)C(t)dt + R(T)B(T)$$

Taking the limit of both sides as $T \to \infty$ and imposing the NPG constraint

$$\lim_{T\to\infty} R(T)B(T) \geq 0$$

then implies that the consumer faces an infinite-horizon present value budget constraint

$$B(0) + \int_0^\infty R(t)W(t)dt \geq \int_0^\infty R(t)C(t)dt$$

NPG and TVC

Then one can argue that unless this infinite-horizon present value budget constraint binds, implying

$$B(0) + \int_0^\infty R(t)W(t)dt = \int_0^\infty R(t)C(t)dt$$

the consumer is not spending enough and overaccumulating assets. The TVC

$$\lim_{T\to\infty} R(T)B(T) = \lim_{T\to\infty} \exp\left[-\int_0^T r(s)ds\right] B(T) = 0$$

is therefore a necessary condition for optimality.

NPG and TVC

Once again, the NPG constraint

$$\lim_{T\to\infty} R(T)B(T) = \lim_{T\to\infty} \exp\left[-\int_0^T r(s)ds\right] B(T) \geq 0$$

is a constraint imposed on the consumer, ruling out schemes in which debt grows without limit.

Whereas the TVC

$$\lim_{T\to\infty} R(T)B(T) = \lim_{T\to\infty} \exp\left[-\int_0^T r(s)ds\right] B(T) = 0$$

is a necessary condition for consumer optimality: if it doesn't hold, the consumer is overaccumulating assets.