The Envelope Theorem

A firm hires n workers to produce output y according to

$$y = n^{\alpha} \text{ with } 0 < \alpha < 1$$

The firm faces the competitive (real) wage w, and chooses n to maximize profits

$$\max_{n} \; n^{\alpha} - wn$$

The first-order condition is

$$\alpha (n^*)^{\alpha - 1} - w = 0$$

The Envelope Theorem

The first-order condition is

$$\alpha(n^*)^{\alpha-1} - w = 0$$

implies that the firm's labor demand curve is

$$n^*(w) = \left(\frac{w}{\alpha}\right)^{1/(\alpha-1)}$$

Since $\alpha < 1$, this labor demand curve is downward-sloping. In response to an increase in w, the firm will hire fewer workers.

The Envelope Theorem

Now define the profit function

$$\pi(w) = \max_{n} \; n^{\alpha} - wn$$

The profit function answers the question: given w, what's the maximum level of profits the firm can earn?

To evaluate the profit function for any value of w, find $n^*(w)$ and compute the implied level of profits:

$$\pi(w) = [n^*(w)]^{\alpha} - wn^*(w)$$

The Envelope Theorem

The envelope theorem implies that the the profit function

$$\pi(w) = [n^*(w)]^\alpha - wn^*(w)$$

satisfies

$$\pi'(w) = -n^*(w)$$

This is Hotelling's lemma: if the firm has hired 100 workers, and the wage goes up by $1 per year, the firm's profits go down by $100 per year.

The Envelope Theorem

If the firm has hired 100 workers, and the wage goes up by \$1 per year, the firm's profits go down by \$100 per year.

But won't the firm reduce its labor demand when w rises? Didn't we just show that the labor demand curve slopes down?

Shouldn't we account for the dependence of n^* on w when differentiating the profit function:

$$\pi(w) = [n^*(w)]^\alpha - wn^*(w)$$

In fact, the change in w does affect n^*, but the change in n^* does not affect profits!

The Envelope Theorem

Use the chain rule to differentiate

$$\pi(w) = [n^*(w)]^\alpha - wn^*(w)$$

with respect to w:

$$\pi'(w) = \alpha[n^*(w)]^{\alpha-1} n^{*\prime}(w) - n^*(w) - wn^{*\prime}(w)$$

or

$$\pi'(w) = \{\alpha[n^*(w)]^{\alpha-1} - w\} n^{*\prime}(w) - n^*(w)$$

In fact, $n^{*\prime}(w) < 0$, but $\alpha[n^*(w)]^{\alpha-1} - w = 0$. Because $n^*(w)$ is chosen optimally, the gains from adjusting labor supply have already been exhausted. Only the direct effect $-n^*(w)$ remains.