The Constraint Qualification

Theorem (Kuhn-Tucker) Let x^* maximize $F(x)$ subject to $c \geq G(x)$, where F and G are both continuously differentiable. Assume, as well, that $G'(x^*) \neq 0$. Then there exists a value λ^* of λ that, together with x^*, satisfies

$$L_1(x^*, \lambda^*) = F'(x^*) - \lambda^* G'(x^*) = 0 \tag{1}$$

$$L_2(x^*, \lambda^*) = c - G(x^*) \geq 0 \tag{2}$$

$$\lambda^* \geq 0 \tag{3}$$

$$\lambda^*[c - G(x^*)] = 0 \tag{4}$$

The Constraint Qualification

$$\max_x \; x \text{ subject to } 0 \geq x \tag{P1}$$

$$\max_x \; -x^2 \text{ subject to } 0 \geq x \tag{P2}$$

$$\max_x \; x^3 \text{ subject to } 0 \geq x \tag{P3}$$

For each example, $G(x) = x$ and therefore $G'(x) = 1$ for all x. The constraint qualification holds, and all four Kuhn-Tucker conditions are satisfied at $(x^*, \lambda^*) = (0, 0)$.

The Constraint Qualification

$$\max_x \; x \text{ subject to } 0 \geq x \tag{P1}$$

$$\max_x \; -x^2 \text{ subject to } 0 \geq x \tag{P2}$$

$$\max_x \; x^3 \text{ subject to } 0 \geq x \tag{P3}$$

These examples illustrate that the constraint qualification usually holds and that it is often quite easy to verify.

The Constraint Qualification

$$\max_x \; x \text{ subject to } 0 \geq x^3 \tag{P4}$$

$$\max_x \; -x^2 \text{ subject to } 0 \geq x^3 \tag{P5}$$

$$\max_x \; x^3 \text{ subject to } 0 \geq x^3 \tag{P6}$$

For each example, $x^* = 0$, $G(x) = x^3$, and therefore $G'(x^*) = 0$. The constraint qualification fails. For (P4), it is not possible to find a value of λ^* that, together with $x^* = 0$, satisfies the four Kuhn-Tucker conditions. (P4) shows that we do need the constraint qualification in order to apply the results of the KT theorem.

The Constraint Qualification

$$\max_x \; x \text{ subject to } 0 \geq x^3 \tag{P4}$$

$$\max_x \; -x^2 \text{ subject to } 0 \geq x^3 \tag{P5}$$

$$\max_x \; x^3 \text{ subject to } 0 \geq x^3 \tag{P6}$$

For (P5) and (P6), the Kuhn-Tucker conditions are satisfied at $(x^*, \lambda^*) = (0, \lambda^*)$ for any value of $\lambda^* \geq 0$. The assumptions of the theorem do not hold, but the method still "works."

The Constraint Qualification

$$\max_x \; x \text{ subject to } 0 \geq x \tag{P1}$$

$$\max_x \; x \text{ subject to } 0 \geq x^3 \tag{P4}$$

But why, exactly, would anyone write the simple constraint from (P1) in the more complicated form from (P4)? Is the problem with the math or with the person using the math?