ECON 772001
MATH FOR ECONOMISTS

Peter Ireland

Boston College

December 1, 2020

The Kuhn-Tucker Theorem Under Uncertainty

The problem: Given (y_0, ε_0), choose contingency plans for z_t, $t = 0, 1, 2, \ldots$, and y_t, $t = 1, 2, 3, \ldots$, to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t F(y_t, z_t, \varepsilon_t)$$

subject to

$$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1}$$

for all $t = 0, 1, 2, \ldots$ and ε_{t+1}.

The Kuhn-Tucker Theorem Under Uncertainty

$H_t = \{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_t\}$ = history of shocks through t

$p(H_t)$ = probability of H_t
(probability mass replaced by density if ε_{t+1} is continuous)

$y_t(H_t)$ = stock variable at t following history H_t

$z_t(H_t)$ = stock variable during t following history H_t

$\varepsilon_t(H_t)$ = shock ε_t at the end of H_t

The notation shows that y_t and z_t are stochastic processes adapted to the information sets generated by $\{\varepsilon_t\}_{t=0}^{\infty}$.

The Kuhn-Tucker Theorem Under Uncertainty

Ω_t = set of all possible H_t

$H_{t+1} = (H_t, \varepsilon_{t+1})$ = history through $t+1$ that follows H_t through t

$\Omega_{t+1}|H_t$ = set of all possible (H_t, ε_{t+1}) that follow H_t

The Kuhn-Tucker Theorem Under Uncertainty

The problem: Given $(y_0(H_0), \varepsilon_0(H_0))$, choose $z_t(H_t)$ for all $t = 0, 1, 2, \ldots$ and $H_t \in \Omega_t$ and $y_t(H_t)$ for all $t = 1, 2, 3, \ldots$ and $H_t \in \Omega_t$ to maximize

$$\sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \beta^t p(H_t) F[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)]$$

subject to

$$y_t(H_t) + Q[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] \geq y_{t+1}(H_t, \varepsilon_{t+1})$$

for all $t = 0, 1, 2, \ldots$, $H_t \in \Omega_t$, and $(H_t, \varepsilon_{t+1}) \in \Omega_{t+1} | H_t$

The Kuhn-Tucker Theorem Under Uncertainty

Choose $z_t(H_t)$ for all $t = 0, 1, \ldots, T$ and $H_t \in \Omega_t$ and $y_t(H_t)$ for all $t = 1, 2, \ldots . T$ and $H_t \in \Omega_t$ to maximize

$$\sum_{t=0}^{T} \sum_{H_t \in \Omega_t} \beta^t p(H_t) F[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)]$$

subject to

$$y_t(H_t) + Q[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] \geq y_{t+1}(H_t, \varepsilon_{t+1})$$

for all $t = 0, 1, \ldots, T$, $H_t \in \Omega_t$, and $(H_t, \varepsilon_{t+1}) \in \Omega_{t+1} | H_t$

Note that with a finite horizon and a finite number of realizations for ε_{t+1}, this problem is a special case of, not an extension to, the problem to which the Kuhn-Tucker theorem applies directly.

The Kuhn-Tucker Theorem Under Uncertainty

$$
\begin{aligned}
L = & \sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \beta^t p(H_t) F[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)] \\
& + \sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1} | H_t} \tilde{\mu}_{t+1}(H_t, \varepsilon_{t+1}) \\
& \times \{ y_t(H_t) + Q[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] - y_{t+1}(H_t, \varepsilon_{t+1}) \}
\end{aligned}
$$

The Kuhn-Tucker Theorem Under Uncertainty

$$L = \sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \beta^t p(H_t) F[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)]$$

$$+ \sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1} | H_t} \tilde{\mu}_{t+1}(H_t, \varepsilon_{t+1})$$

$$\times \{y_t(H_t) + Q[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] - y_{t+1}(H_t, \varepsilon_{t+1})\}$$

Here, the Lagrangian is in "present value" form. To convert to "current values" use the change of variables

$$\tilde{\mu}_{t+1}(H_t, \varepsilon_{t+1}) = \beta^{t+1} p(H_t, \varepsilon_{t+1}) \mu_{t+1}(H_t, \varepsilon_{t+1})$$

The Kuhn-Tucker Theorem Under Uncertainty

$$
\begin{aligned}
L = &\sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \beta^t p(H_t) F[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)] \\
&+ \sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1} | H_t} \tilde{\mu}_{t+1}(H_t, \varepsilon_{t+1}) \\
&\times \{y_t(H_t) + Q[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] - y_{t+1}(H_t, \varepsilon_{t+1})\}
\end{aligned}
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\begin{aligned}
L = &\sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \beta^t p(H_t) F[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)] \\
&+ \sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1} | H_t} \beta^{t+1} p(H_t, \varepsilon_{t+1}) \mu_{t+1}(H_t, \varepsilon_{t+1}) \\
&\times \{y_t(H_t) + Q[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] - y_{t+1}(H_t, \varepsilon_{t+1})\}
\end{aligned}
$$

The Kuhn-Tucker Theorem Under Uncertainty

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\begin{aligned}
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& + \sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1} | H_t} \beta^{t+1} p(H_t, \varepsilon_{t+1}) \mu_{t+1}(H_t, \varepsilon_{t+1}) \\
& \times \{y_t(H_t) + Q[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] - y_{t+1}(H_t, \varepsilon_{t+1})\}
\end{aligned}
$$

Fix $t = 0, 1, 2, \ldots,$ and $H_t \in \Omega_t$. FOC for $z_t(H_t)$:

$$
\begin{aligned}
& \beta^t p(H_t) F_2[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)] \\
& + \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1} | H_t} \beta^{t+1} p(H_t, \varepsilon_{t+1}) \mu_{t+1}(H_t, \varepsilon_{t+1}) \\
& \times Q_2[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] = 0
\end{aligned}
$$

The Kuhn-Tucker Theorem Under Uncertainty

FOC for $z_t(H_t)$:

$$\beta^t p(H_t) F_2[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)]$$
$$+ \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1} | H_t} \beta^{t+1} p(H_t, \varepsilon_{t+1}) \mu_{t+1}(H_t, \varepsilon_{t+1})$$
$$\times Q_2[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] = 0$$

Recall that

$$\frac{p(H_t, \varepsilon_{t+1})}{p(H_t)} = p(\varepsilon_{t+1} | H_t)$$

The Kuhn-Tucker Theorem Under Uncertainty

$$\beta^t p(H_t) F_2[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)]$$
$$+ \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1} | H_t} \beta^{t+1} p(H_t, \varepsilon_{t+1}) \mu_{t+1}(H_t, \varepsilon_{t+1})$$
$$\times Q_2[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] = 0$$

$$F_2[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)]$$
$$+ \beta \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1} | H_t} \beta^{t+1} p(\varepsilon_{t+1} | H_t) \mu_{t+1}(H_t, \varepsilon_{t+1})$$
$$\times Q_2[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] = 0$$

The Kuhn-Tucker Theorem Under Uncertainty

$$
\begin{aligned}
& F_2[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)] \\
& + \beta \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1} | H_t} \beta^{t+1} p(\varepsilon_{t+1} | H_t) \mu_{t+1}(H_t, \varepsilon_{t+1}) \\
& \times Q_2[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] = 0
\end{aligned}
$$

$$
\begin{aligned}
& F_2[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)] \\
& + \beta E_t[\mu_{t+1}(H_t, \varepsilon_{t+1}) Q_2[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] = 0
\end{aligned}
$$

The Kuhn-Tucker Theorem Under Uncertainty

$$F_2[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)] \\ + \beta E_t[\mu_{t+1}(H_t, \varepsilon_{t+1}) Q_2[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] = 0$$

$$F_2(y_t, z_t, \varepsilon_t) + \beta E_t[\mu_{t+1} Q_2(y_t, z_t, \varepsilon_{t+1})] = 0$$

Coincides with (23), with $\mu_{t+1} = v_1(y_{t+1}, \varepsilon_{t+1})$.

The Kuhn-Tucker Theorem Under Uncertainty

$$L = \sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \beta^t p(H_t) F[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)]$$
$$+ \sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1} | H_t} \beta^{t+1} p(H_t, \varepsilon_{t+1}) \mu_{t+1}(H_t, \varepsilon_{t+1})$$
$$\times \{y_t(H_t) + Q[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] - y_{t+1}(H_t, \varepsilon_{t+1})\}$$

Fix $t = 1, 2, 3, \ldots,$ and $H_t \in \Omega_t$. FOC for $y_t(H_t)$:

$$\beta^t p(H_t) F_1[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)]$$
$$+ \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1} | H_t} \beta^{t+1} p(H_t, \varepsilon_{t+1}) \mu_{t+1}(H_t, \varepsilon_{t+1})$$
$$\times \{1 + Q_1[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})]\}$$
$$- \beta^t p(H_t) \mu_t(H_t) = 0$$

The Kuhn-Tucker Theorem Under Uncertainty

$$\begin{aligned}
&\beta^t p(H_t) F_1[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)] \\
&+ \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1} | H_t} \beta^{t+1} p(H_t, \varepsilon_{t+1}) \mu_{t+1}(H_t, \varepsilon_{t+1}) \\
&\times \{1 + Q_1[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})]\} \\
&- \beta^t p(H_t) \mu_t(H_t) = 0
\end{aligned}$$

$$\begin{aligned}
\mu_t(H_t) = {} & F_1[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)] \\
& + \beta \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1} | H_t} p(\varepsilon_{t+1} | H_t) \mu_{t+1}(H_t, \varepsilon_{t+1}) \\
& \times \{1 + Q_1[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})]\}
\end{aligned}$$

The Kuhn-Tucker Theorem Under Uncertainty

$$\begin{aligned}\mu_t(H_t) = {} & F_1[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)] \\ & + \beta \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1} | H_t} p(\varepsilon_{t+1} | H_t) \mu_{t+1}(H_t, \varepsilon_{t+1}) \\ & \times \{1 + Q_1[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})]\}\end{aligned}$$

$$\begin{aligned}\mu_t(H_t) = {} & F_1[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)] \\ & + \beta E_t \left[\mu_{t+1}(H_t, \varepsilon_{t+1}) \{1 + Q_1[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})]\}\right)\right]\end{aligned}$$

The Kuhn-Tucker Theorem Under Uncertainty

$$\mu_t(H_t) = F_1[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)]$$
$$+ \beta E_t \left[\mu_{t+1}(H_t, \varepsilon_{t+1})\{1 + Q_1[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})]\}\right)]$$

$$\mu_t = F_1(y_t, z_t, \varepsilon_t) + \beta E_t\{\mu_{t+1}[1 + Q_1(y_t, z_t, \varepsilon_{t+1})]\}$$

Coincides with (24), with $\mu_t = v_1(y_t, \varepsilon_t)$ and $\mu_{t+1} = v_1(y_{t+1}, \varepsilon_{t+1})$.

The Kuhn-Tucker Theorem Under Uncertainty

$$F_2(y_t, z_t, \varepsilon_t) + \beta E_t[\mu_{t+1} Q_2(y_t, z_t, \varepsilon_{t+1})] = 0$$

$$\mu_t = F_1(y_t, z_t, \varepsilon_t) + \beta E_t\{\mu_{t+1}[1 + Q_1(y_t, z_t, \varepsilon_{t+1})]\}$$

Together with the binding constraint

$$y_{t+1} = y_t + Q(y_t, z_t, \varepsilon_{t+1})$$

and a law of motion for ε_t, the FOCs form a system of 4 equations in 4 unknowns: y_t, z_t, μ_t, and ε_t.