ECON 772001
MATH FOR ECONOMISTS

Peter Ireland

Boston College

November 24, 2020

Saving with Random Returns

Extends the previous life-cycle saving example by allowing the consumer to invest in $n \geq 1$ assets with risky (random) returns.

$A_t =$ beginning-of-period financial assets

$A_0 > 0$, so that consumption can be financed out of capital income alone (no labor income)

$c_t =$ consumption

Saving with Random Returns

s_{it} = savings allocated to each asset $i = 1, 2, \ldots, n$

$$A_t \geq c_t + \sum_{i=1}^{n} s_{it}$$

for all $t = 0, 1, 2, \ldots$.

Saving with Random Returns

R_{it+1} = gross random return on asset i between t and $t+1$

When s_{it} is chosen:

R_{it} is known
R_{it+1} is random

$R_{t+1} = \begin{bmatrix} R_{1t+1} & R_{2t+1} & \dots & R_{nt+1} \end{bmatrix}$ = vector of returns

The returns in R_{t+1} can be mutually and serially correlated. They need not be normally distributed. The only restriction is that they have the Markov property.

Hence, the theory applies to stock, bonds, options, other derivatives, commodities, foreign exchange . . . .

Saving with Random Returns

$$A_{t+1} = \sum_{i=1}^{n} R_{it+1} s_{it}$$

Or, allowing for free disposal,

$$\sum_{i=1}^{n} R_{it+1} s_{it} \geq A_{t+1}$$

for all $t = 0, 1, 2, \ldots$ and R_{t+1}

Expected utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) = E_0 \sum_{t=0}^{\infty} \beta^t u\left(A_t - \sum_{i=1}^{n} s_{it}\right)$$

Saving with Random Returns

The problem: Given $A_0 > 0$ and R_0, choose contingency plans for s_{it}, $i = 1, 2, \ldots, n$ and $t = 0, 1, 2, \ldots$ and A_t, $t = 1, 2, 3, \ldots$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u \left(A_t - \sum_{i=1}^{n} s_{it} \right)$$

subject to

$$\sum_{i=1}^{n} R_{it+1} s_{it} \geq A_{t+1}$$

for all $t = 0, 1, 2, \ldots$ and R_{t+1}

Saving with Random Returns

Choose contingency plans for s_{it} and A_t to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(A_t - \sum_{i=1}^{n} s_{it}\right)$$

subject to

$$\sum_{i=1}^{n} R_{it+1} s_{it} \geq A_{t+1} \text{ for all } t = 0, 1, 2, \ldots \text{ and } R_{t+1}$$

The Bellman equation:

$$v(A_t, R_t) = \max_{\{s_{it}\}_{i=1}^{n}} u\left(A_t - \sum_{i=1}^{n} s_{it}\right) + \beta E_t v\left(\sum_{i=1}^{n} R_{it+1} s_{it}, R_{t+1}\right)$$

Saving with Random Returns

$$v(A_t, R_t) = \max_{\{s_{it}\}_{i=1}^n} u\left(A_t - \sum_{i=1}^n s_{it}\right) + \beta E_t v\left(\sum_{i=1}^n R_{it+1} s_{it}, R_{t+1}\right)$$

FOC for s_{it}:

$$-u'\left(A_t - \sum_{i=1}^n s_{it}\right) + \beta E_t\left[R_{it+1} v_1\left(\sum_{i=1}^n R_{it+1} s_{it}, R_{t+1}\right)\right] = 0$$

for all $i = 1, 2, \ldots, n$ and $t = 0, 1, 2, \ldots$

Saving with Random Returns

$$v(A_t, R_t) = \max_{\{s_{it}\}_{i=1}^n} u\left(A_t - \sum_{i=1}^n s_{it}\right) + \beta E_t v\left(\sum_{i=1}^n R_{it+1} s_{it}, R_{t+1}\right)$$

Envelope condition for A_t:

$$v_1(A_t, R_t) = u'\left(A_t - \sum_{i=1}^n s_{it}\right)$$

for all $t = 1, 2, 3, \ldots$

Saving with Random Returns

FOC for s_{it}:

$$-u'\left(A_t - \sum_{i=1}^{n} s_{it}\right) + \beta E_t\left[R_{it+1} v_1\left(\sum_{i=1}^{n} R_{it+1} s_{it}, R_{t+1}\right)\right] = 0$$

Use the constraints to simplify:

$$u'(c_t) = \beta E_t\left[R_{it+1} v_1(A_{t+1}, R_{t+1})\right]$$

for all $i = 1, 2, \ldots, n$ and $t = 0, 1, 2, \ldots$

Saving with Random Returns

Envelope condition for A_t:

$$v_1(A_t, R_t) = u'\left(A_t - \sum_{i=1}^{n} s_{it}\right)$$

Use the constraints to simplify:

$$v_1(A_t, R_t) = u'(c_t)$$

for all $t = 1, 2, 3, \ldots$

Saving with Random Returns

Combine

$$u'(c_t) = \beta E_t\left[R_{it+1} v_1(A_{t+1}, R_{t+1})\right]$$

for all $i = 1, 2, \ldots, n$ and $t = 0, 1, 2, \ldots$ and

$$v_1(A_t, R_t) = u'(c_t)$$

for all $t = 1, 2, 3, \ldots$ to obtain

$$u'(c_t) = \beta E_t\left[R_{it+1} u'(c_{t+1})\right] \tag{26}$$

for all $i = 1, 2, \ldots, n$ and $t = 0, 1, 2, \ldots$

Saving with Random Returns

$$u'(c_t) = \beta E_t \left[R_{it+1} u'(c_{t+1}) \right] \tag{26}$$

for all $i = 1, 2, \dots, n$ and $t = 0, 1, 2, \dots$

Generalizes (19) to the case of uncertainty.

It must hold for *all* asset returns.

Saving with Random Returns

Let

$$m_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)} = \text{ IMRS}$$

denote the intertemporal marginal rate of substitution.

Rewrite

$$u'(c_t) = \beta E_t \left[R_{it+1} u'(c_{t+1}) \right] \tag{26}$$

more compactly as

$$1 = E_t(R_{it+1} m_{t+1}) \tag{27}$$

for all $i = 1, 2, \ldots, n$ and $t = 0, 1, 2, \ldots$

Saving with Random Returns

Now assume there is a risk-free asset, with return R^f_{t+1} between t and $t+1$ that is known at t.

For this asset,

$$1 = E_t(R_{it+1} m_{t+1}) \tag{27}$$

implies

$$1 = R^f_{t+1} E_t(m_{t+1})$$

or

$$E_t(m_{t+1}) = \frac{1}{R^f_{t+1}} \tag{28}$$

Saving with Random Returns

And recall that for any two random variables X and Y:

$$\begin{aligned}
\operatorname{cov}(X, Y) &= E\{[X - E(X)][Y - E(Y)]\} \\
&= E[XY - E(X)Y - XE(Y) + E(X)E(Y)] \\
&= E(XY) - E(X)E(Y) - E(X)E(Y) + E(X)E(Y) \\
&= E(XY) - E(X)E(Y)
\end{aligned}$$

$$E(XY) = E(X)E(Y) + \operatorname{cov}(X, Y)$$

Saving with Random Returns

Using (28) and this fact about covariance,

$$1 = E_t(R_{it+1}m_{t+1}) \tag{27}$$

implies

$$1 = E_t(R_{it+1})E_t(m_{t+1}) + \text{cov}_t(R_{it+1}, m_{t+1})$$

$$1 = E_t(R_{it+1})\left(\frac{1}{R^f_{t+1}}\right) + \text{cov}_t(R_{it+1}, m_{t+1})$$

$$R^f_{t+1} = E_t(R_{it+1}) + R^f_{t+1}\text{cov}_t(R_{it+1}, m_{t+1})$$

$$E_t(R_{it+1}) - R^f_{t+1} = -R^f_{t+1}\text{cov}_t(R_{it+1}, m_{t+1}) \tag{29}$$

Saving with Random Returns

$$E_t(R_{it+1}) - R^f_{t+1} = -R^f_{t+1}\text{cov}_t(R_{it+1}, m_{t+1}) \qquad (29)$$

The expected excess return on asset i has a sign opposite to that of the covariance between R_{it+1} and m_{t+1}.

Saving with Random Returns

$$E_t(R_{it+1}) - R^f_{t+1} = -R^f_{t+1}\text{cov}_t(R_{it+1}, m_{t+1}) \qquad (29)$$

The typical stock has a high return R_{it+1} during good economic times, when the IMRS

$$m_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)}$$

is low.

For most stocks, therefore, $\text{cov}_t(R_{it+1}, m_{t+1}) < 0$ and the theory correctly predicts that $E_t(R_{it+1}) > R^f_{t+1}$.

Saving with Random Returns

$$E_t(R_{it+1}) - R^f_{t+1} = -R^f_{t+1}\text{cov}_t(R_{it+1}, m_{t+1}) \qquad (29)$$

An asset with a high return R_{it+1} during bad economic times, when the IMRS

$$m_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)}$$

is high, has $\text{cov}_t(R_{it+1}, m_{t+1}) > 0$ and hence, according to the theory, $E_t(R_{it+1}) < R^f_{t+1}$.

Investors will hold this asset (maybe gold, or long-term nominal US government bonds), despite its low expected return, because it provides macroeconomic insurance.

Saving with Random Returns

$$E_t(R_{it+1}) - R^f_{t+1} = -R^f_{t+1}\text{cov}_t(R_{it+1}, m_{t+1}) \qquad (29)$$

The problem with this theory is quantitative: since consumption is so smooth (ironically, because the PIH actually works reasonably well at least at the aggregate level), $\text{cov}_t(R_{it+1}, m_{t+1})$ is generally too small to explain large differences in expected returns across assets.

Saving with Random Returns

Rajnish Mehra and Edward C. Prescott. "The Equity Premium: A Puzzle." *Journal of Monetary Economics* 1985.

Lars Peter Hansen and Kenneth J. Singleton. "Generalized Instrumental Variables Estimation of Nonlinear Rational Expectations Models." *Econometrica* 1982.

Lars Peter Hansen and Kenneth J. Singleton. "Stochastic Consumption, Risk Aversion, and the Temporal Behavior of Asset Returns." *Journal of Political Economy* 1983.

Lars Peter Hansen and Ravi Jagannathan. "Implications of Security Market Data for Models of Dynamic Economies." *Journal of Political Economy* 1991.

Saving with Random Returns

John Y. Campbell and John H. Cochrane. "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior." *Journal of Political Economy* 1999.

Ravi Bansal and Amir Yaron. "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles." *Journal of Finance* 2004.