ECON 772001
MATH FOR ECONOMISTS

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November 19, 2020

A Dynamic Stochastic Problem

The problem: Given (y_0, ε_0), choose contingency plans for z_t, $t = 0, 1, 2, \ldots$, and y_t, $t = 1, 2, 3, \ldots$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t F(y_t, z_t, \varepsilon_t)$$

subject to

$$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1}$$

for all $t = 0, 1, 2, \ldots$ and all possible realizations of ε_{t+1}.

Stochastic Dynamic Programming

At $t = 0$, given (y_0, ε_0), define the value function

$$v(y_0, \varepsilon_0) = \max_{\{z_t\}_{t=0}^{\infty}, \{y_t\}_{t=1}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t F(y_t, z_t, \varepsilon_t)$$

subject to

$$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1}$$

for all $t = 0, 1, 2, \ldots$ and ε_{t+1}

Stochastic Dynamic Programming

At $t = 0, 1, 2, \ldots$, given (y_t, ε_t), define the value function

$$v(y_t, \varepsilon_t) = \max_{\{z_{t+j}\}_{j=0}^{\infty}, \{y_{t+j}\}_{j=1}^{\infty}} E_t \sum_{j=0}^{\infty} \beta^j F(y_{t+j}, z_{t+j}, \varepsilon_{t+j})$$

subject to

$$y_{t+j} + Q(y_{t+j}, z_{t+j}, \varepsilon_{t+j+1}) \geq y_{t+j+1}$$

for all $j = 0, 1, 2, \ldots$ and ε_{t+j+1}

Stochastic Dynamic Programming

Separate out the time-t components:

$$v(y_t, \varepsilon_t) = \max_{z_t, y_{t+1}} \left[F(y_t, z_t, \varepsilon_t) + \max_{\{z_{t+j}\}_{j=1}^{\infty}, \{y_{t+j}\}_{j=2}^{\infty}} E_t \sum_{j=1}^{\infty} \beta^j F(y_{t+j}, z_{t+j}, \varepsilon_{t+j}) \right]$$

subject to

$$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1} \text{ for all } \varepsilon_{t+1}$$

$$y_{t+j} + Q(y_{t+j}, z_{t+j}, \varepsilon_{t+j+1}) \geq y_{t+j+1}$$

for all $j = 1, 2, 3, \ldots$ and ε_{t+j+1}

Stochastic Dynamic Programming

$$v(y_t, \varepsilon_t) = \max_{z_t, y_{t+1}} \left[F(y_t, z_t, \varepsilon_t) + \max_{\{z_{t+j}\}_{j=1}^{\infty}, \{y_{t+j}\}_{j=2}^{\infty}} E_t \sum_{j=1}^{\infty} \beta^j F(y_{t+j}, z_{t+j}, \varepsilon_{t+j}) \right]$$

subject to

$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1}$ for all ε_{t+1}

$y_{t+j} + Q(y_{t+j}, z_{t+j}, \varepsilon_{t+j+1}) \geq y_{t+j+1}$

for all $j = 1, 2, 3, \ldots$ and ε_{t+j+1}

Relable the time indices:

$$v(y_t, \varepsilon_t) = \max_{z_t, y_{t+1}} \left[F(y_t, z_t, \varepsilon_t) + \beta \max_{\{z_{t+1+j}\}_{j=0}^{\infty}, \{y_{t+1+j}\}_{j=1}^{\infty}} E_t \sum_{j=0}^{\infty} \beta^j F(y_{t+1+j}, z_{t+1+j}, \varepsilon_{t+1+j}) \right]$$

subject to

$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1}$ for all ε_{t+1}

$y_{t+1+j} + Q(y_{t+1+j}, z_{t+1+j}, \varepsilon_{t+j+1}) \geq y_{t+1+j+1}$

for all $j = 0, 1, 2, \ldots$ and $\varepsilon_{t+1+j+1}$

Stochastic Dynamic Programming

Law of iterated expectations: For any random variable X_{t+j}:

$$E_t[E_{t+1}(X_{t+j})] = E_t(X_{t+j})$$

Example: AR(1)

$$\varepsilon_{t+1} = \rho\varepsilon_t + \eta_{t+1},\ \eta_{t+1} \sim N(0, \sigma^2)$$

$$\varepsilon_{t+2} = \rho\varepsilon_{t+1} + \eta_{t+2} = \rho^2\varepsilon_t + \rho\eta_{t+1} + \eta_{t+2}$$

$$E_{t+1}(\varepsilon_{t+2}) = E_{t+1}(\rho^2\varepsilon_t + \rho\eta_{t+1} + \eta_{t+2}) = \rho^2\varepsilon_t + \rho\eta_{t+1}$$

$$E_t[E_{t+1}(\varepsilon_{t+2})] = E_t(\rho^2\varepsilon_t + \rho\eta_{t+1}) = \rho^2\varepsilon_t$$

$$E_t(\varepsilon_{t+2}) = E_t(\rho\varepsilon_{t+1} + \eta_{t+2}) = E_t(\rho^2\varepsilon_t + \rho\eta_{t+1} + \eta_{t+2}) = \rho^2\varepsilon_t$$

Stochastic Dynamic Programming

$$v(y_t, \varepsilon_t) = \max_{z_t, y_{t+1}} \left[F(y_t, z_t, \varepsilon_t) + \beta \max_{\{z_{t+1+j}\}_{j=0}^{\infty}, \{y_{t+1+j}\}_{j=1}^{\infty}} E_t \sum_{j=0}^{\infty} \beta^j F(y_{t+1+j}, z_{t+1+j}, \varepsilon_{t+1+j}) \right]$$

subject to

$$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1} \text{ for all } \varepsilon_{t+1}$$

$$y_{t+1+j} + Q(y_{t+1+j}, z_{t+1+j}, \varepsilon_{t+j+1}) \geq y_{t+1+j+1}$$

for all $j = 0, 1, 2, \ldots$ and $\varepsilon_{t+1+j+1}$

$$v(y_t, \varepsilon_t) = \max_{z_t, y_{t+1}} \left[F(y_t, z_t, \varepsilon_t) + \beta \max_{\{z_{t+1+j}\}_{j=0}^{\infty}, \{y_{t+1+j}\}_{j=1}^{\infty}} E_t E_{t+1} \sum_{j=0}^{\infty} \beta^j F(y_{t+1+j}, z_{t+1+j}, \varepsilon_{t+1+j}) \right]$$

subject to

$$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1} \text{ for all } \varepsilon_{t+1}$$

$$y_{t+1+j} + Q(y_{t+1+j}, z_{t+1+j}, \varepsilon_{t+j+1}) \geq y_{t+1+j+1}$$

for all $j = 0, 1, 2, \ldots$ and $\varepsilon_{t+1+j+1}$

Stochastic Dynamic Programming

$$v(y_t, \varepsilon_t) = \max_{z_t, y_{t+1}} \left[F(y_t, z_t, \varepsilon_t) + \beta \max_{\{z_{t+1+j}\}_{j=0}^{\infty}, \{y_{t+1+j}\}_{j=1}^{\infty}} E_t E_{t+1} \sum_{j=0}^{\infty} \beta^j F(y_{t+1+j}, z_{t+1+j}, \varepsilon_{t+1+j}) \right]$$

subject to

$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1}$ for all ε_{t+1}

$y_{t+1+j} + Q(y_{t+1+j}, z_{t+1+j}, \varepsilon_{t+j+1}) \geq y_{t+1+j+1}$

for all $j = 0, 1, 2, \ldots$ and $\varepsilon_{t+1+j+1}$

Since the terms in red define $v(y_{t+1}, \varepsilon_{t+1})$:

$$v(y_t, \varepsilon_t) = \max_{z_t, y_{t+1}} [F(y_t, z_t, \varepsilon_t) + \beta E_t v(y_{t+1}, \varepsilon_{t+1})]$$

subject to

$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1}$ for all ε_{t+1}

Stochastic Dynamic Programming

$$v(y_t, \varepsilon_t) = \max_{z_t, y_{t+1}} [F(y_t, z_t, \varepsilon_t) + \beta E_t v(y_{t+1}, \varepsilon_{t+1})]$$

$$\text{subject to}$$

$$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1} \text{ for all } \varepsilon_{t+1}$$

If the constraint always binds:

$$v(y_t, \varepsilon_t) = \max_{z_t} F(y_t, z_t, \varepsilon_t) + \beta E_t v[y_t + Q(y_t, z_t, \varepsilon_{t+1}), \varepsilon_{t+1}] \tag{22}$$

Stochastic Dynamic Programming

$$v(y_t, \varepsilon_t) = \max_{z_t} F(y_t, z_t, \varepsilon_t) + \beta E_t v[y_t + Q(y_t, z_t, \varepsilon_{t+1}), \varepsilon_{t+1}] \quad (22)$$

Equation (22) is the Bellman equation for the stochastic case.

The optimization problem in (22) has a single choice *variable* z_t, and does not explicitly involve the choice of a stochastic process.

Stochastic Dynamic Programming

$$v(y_t, \varepsilon_t) = \max_{z_t} F(y_t, z_t, \varepsilon_t) + \beta E_t v[y_t + Q(y_t, z_t, \varepsilon_{t+1}), \varepsilon_{t+1}] \tag{22}$$

FOC for z_t:

$$F_2(y_t, z_t, \varepsilon_t) + \beta E_t\{v_1[y_t + Q(y_t, z_t, \varepsilon_{t+1}), \varepsilon_{t+1}]Q_2(y_t, z_t, \varepsilon_{t+1})\} = 0 \tag{23}$$

If we had used the Lagrangian instead,

$$\mu_{t+1}(y_{t+1}, \varepsilon_{t+1}) = v_1[y_t + Q(y_t, z_t, \varepsilon_{t+1}), \varepsilon_{t+1}]$$

Stochastic Dynamic Programming

$$v(y_t, \varepsilon_t) = \max_{z_t} F(y_t, z_t, \varepsilon_t) \\ + \beta E_t v[y_t + Q(y_t, z_t, \varepsilon_{t+1}), \varepsilon_{t+1}] \quad (22)$$

Envelope condition for y_t:

$$v_1(y_t, \varepsilon_t) \\ = F_1(y_t, z_t, \varepsilon_t) \\ + \beta E_t\{v_1[y_t + Q(y_t, z_t, \varepsilon_{t+1}), \varepsilon_{t+1}][1 + Q_1(y_t, z_t, \varepsilon_{t+1})\} = 0 \quad (24)$$

If we had used the Lagrangian instead,

$$\mu_t(y_t, \varepsilon_t) = v_1(y_t, \varepsilon_t)$$

$$\mu_{t+1}(y_{t+1}, \varepsilon_{t+1}) = v_1[y_t + Q(y_t, z_t, \varepsilon_{t+1}), \varepsilon_{t+1}]$$

Stochastic Dynamic Programming

Together with the binding constraint

$$y_{t+1} = y_t + Q(y_t, z_t, \varepsilon_{t+1}) \tag{25}$$

and a law of motion for ε_{t+1} (for example, an AR(1)), (23) and (24) form a system of 4 equations in 4 unknowns: y_t, z_t, ε_t, and $v(y_t, \varepsilon_t) = \mu_t$.