ECON 772001
MATH FOR ECONOMISTS

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Stochastic Dynamic Programming

A Dynamic Stochastic Optimization Problem

The Bellman Equation

Example 3: Saving with Random Returns

Method of Lagrange Multipliers

A Dynamic Stochastic Problem

Discrete time, infinite horizon: $t = 0, 1, 2, \ldots$

y_t = state variable

z_t = control variable

ε_{t+1} = random shock

z_t is chosen after ε_t becomes known, but before ε_{t+1} is realized

A Dynamic Stochastic Problem

The shock ε_{t+1} can be serially correlated, but must have the *Markov property*: its distribution can depend on ε_t, but not on $\varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, \ldots$

Example: AR(1)

$$\varepsilon_{t+1} = \rho\varepsilon_t + \eta_{t+1}, \text{ where } \eta_{t+1} \sim N(0, \sigma^2)$$

A Dynamic Stochastic Problem

Another example: AR(2)

$$\varepsilon_{t+1} = \rho_0 \varepsilon_t + \rho_1 \varepsilon_{t-1} + \eta_{t+1}$$

$$\begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_t \end{bmatrix} = \begin{bmatrix} \rho_0 & \rho_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \varepsilon_{t-1} \end{bmatrix} + \begin{bmatrix} \eta_{t+1} \\ 0 \end{bmatrix}$$

$$X_{t+1} = PX_t + \tilde{\eta}_{t+1}$$

We can allow for additional lags, at the cost of having to keep track of a longer random vector X_{t+1}.

A Dynamic Stochastic Problem

Objective function

$$E_0 \sum_{t=0}^{\infty} \beta^t F(y_t, z_t, \varepsilon_t)$$

where E_0 denotes the expected value at $t = 0$.

Constraints:

$$Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1} - y_t$$

for all $t = 0, 1, 2, \ldots$ and all possible realizations of ε_{t+1}.

A Dynamic Stochastic Problem

At $t = 0$, $I_0 = \{y_0, \varepsilon_0\}$ are known. z_0 gets chosen

ε_1 is realized and $y_1 = y_0 + Q(y_0, z_0, \varepsilon_1)$ determined

At $t = 1$, $I_1 = \{y_1.\varepsilon_1, y_0, \varepsilon_0\}$ are known. z_1 gets chosen

ε_2 is realized and $y_2 = y_1 + Q(y_1, z_1, \varepsilon_2)$ determined

$\{z_t\}_{t=0}^{\infty}$ and $\{y_t\}_{t=1}^{\infty}$ are themselves stochastic processes, "adapted" to the expanding information sets $\{I_t\}_{t=0}^{\infty}$ "generated" by $\{\varepsilon_t\}_{t=0}^{\infty}$

A Dynamic Stochastic Problem

The recursive form of the constraints and the Markov structure of the shocks imply, however, that for any random variable X_{t+j}:

$$\begin{aligned} E_t(X_{t+j}) &= E(X_{t+j}|I_t) \\ &= E(X_{t+j}|y_t, \varepsilon_t, y_{t-1}, \varepsilon_{t-1}, \ldots, y_0, \varepsilon_0) \\ &= E(X_{t+j}|y_t, \varepsilon_t) \end{aligned}$$

Therefore, (y_t, ε_t) completely summarize the state at t.

A Dynamic Stochastic Problem

The problem: Given (y_0, ε_0), choose contingency plans for z_t, $t = 0, 1, 2, \ldots$, and y_t, $t = 1, 2, 3, \ldots$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t F(y_t, z_t, \varepsilon_t)$$

subject to

$$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1}$$

for all $t = 0, 1, 2, \ldots$ and all possible realizations of ε_{t+1}.

Stochastic Dynamic Programming

At $t = 0$, given (y_0, ε_0), define the value function

$$v(y_0, \varepsilon_0) = \max_{\{z_t\}_{t=0}^{\infty}, \{y_t\}_{t=1}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t F(y_t, z_t, \varepsilon_t)$$

subject to

$$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1}$$

for all $t = 0, 1, 2, \ldots$ and ε_{t+1}

Stochastic Dynamic Programming

At $t = 0, 1, 2, \ldots$, given (y_t, ε_t), define the value function

$$v(y_t, \varepsilon_t) = \max_{\{z_{t+j}\}_{j=0}^{\infty}, \{y_{t+j}\}_{j=1}^{\infty}} E_t \sum_{j=0}^{\infty} \beta^j F(y_{t+j}, z_{t+j}, \varepsilon_{t+j})$$

subject to

$$y_{t+j} + Q(y_{t+j}, z_{t+j}, \varepsilon_{t+j+1}) \geq y_{t+j+1}$$

for all $j = 0, 1, 2, \ldots$ and ε_{t+j+1}