ECON 772001
MATH FOR ECONOMISTS

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Saving Under Certainty

A_t = beginning-of-period assets (bank account balance)

A_0 given

A_t can be negative: borrowing is allowed

But borrowing will be limited by a no-Ponzi-game constraint

Saving Under Certainty

y_t = labor income (exogenous, deterministic)

c_t = consumption

$s_t = A_t + y_t - c_t$ = gross savings

r = constant interest rate

$$(1 + r)s_t \geq A_{t+1}$$

for all $t = 0, 1, 2, \ldots$

Saving Under Certainty

Utility

$$\sum_{t=0}^{\infty} \beta^t u(c_t) = \sum_{t=0}^{\infty} \beta^t u(A_t + y_t - s_t)$$

with $u' > 0$ and $u'' < 0$.

The problem: Given A_0, choose sequences $\{s_t\}_{t=0}^{\infty}$ and $\{A_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(A_t + y_t - s_t)$$

subject to

$$(1+r)s_t \geq A_{t+1}$$

for all $t = 0, 1, 2, \ldots$

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For the problem: Given A_0, choose sequences $\{s_t\}_{t=0}^{\infty}$ and $\{A_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(A_t + y_t - s_t)$$

subject to

$$(1+r)s_t \geq A_{t+1}$$

for all $t = 0, 1, 2, \ldots$ a full description of the state at time t must include $\{y_{t+j}\}_{j=0}^{\infty}$ as well as A_t.

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$$\sum_{t=0}^{\infty} \beta^t u(A_t + y_t - s_t)$$

subject to

$$(1 + r)s_t \geq A_{t+1}$$

for all $t = 0, 1, 2, \ldots$

Bellman equation:

$$v(A_t; \{y_{t+j}\}_{j=0}^{\infty}) = \max_{s_t} \; u(A_t + y_t - s_t) + \beta v((1+r)s_t; \{y_{t+1+j}\}_{j=0}^{\infty})$$

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$$v(A_t; \{y_{t+j}\}_{j=0}^{\infty}) = \max_{s_t} \; u(A_t + y_t - s_t) + \beta v((1+r)s_t; \{y_{t+1+j}\}_{j=0}^{\infty})$$

Or, more simply,

$$v(A_t; t) = \max_{s_t} \; u(A_t + y_t - s_t) + \beta v((1+r)s_t; t+1)$$

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Before characterizing the solution to the problem, let's look more closely at the constraints.

Since $u' > 0$,

$$(1+r)s_t = A_{t+1}$$

$$A_t + y_t - c_t = \left(\frac{1}{1+r}\right) A_{t+1}$$

$$A_t = \left(\frac{1}{1+r}\right) A_{t+1} + c_t - y_t$$

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$$A_t = \left(\frac{1}{1+r}\right) A_{t+1} + c_t - y_t$$

$$A_{t+1} = \left(\frac{1}{1+r}\right) A_{t+2} + c_{t+1} - y_{t+1}$$

$$A_t = \left(\frac{1}{1+r}\right)^2 A_{t+2} + \left(\frac{1}{1+r}\right)(c_{t+1} - y_{t+1}) + c_t - y_t$$

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$$A_t = \left(\frac{1}{1+r}\right)^2 A_{t+2} + \left(\frac{1}{1+r}\right)(c_{t+1} - y_{t+1}) + c_t - y_t$$

$$A_{t+2} = \left(\frac{1}{1+r}\right) A_{t+3} + c_{t+2} - y_{t+2}$$

$$A_t = \left(\frac{1}{1+r}\right)^3 A_{t+3} + \left(\frac{1}{1+r}\right)^2 (c_{t+2} - y_{t+2}) + \left(\frac{1}{1+r}\right)(c_{t+1} - y_{t+1}) + c_t - y_t$$

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$$A_t = \left(\frac{1}{1+r}\right)^3 A_{t+3} + \left(\frac{1}{1+r}\right)^2 (c_{t+2} - y_{t+2}) + \left(\frac{1}{1+r}\right)(c_{t+1} - y_{t+1}) + c_t - y_t$$

Continuing . . .

$$A_t = \left(\frac{1}{1+r}\right)^T A_{t+T} + \sum_{j=0}^{T-1} \left(\frac{1}{1+r}\right)^j (c_{t+j} - y_{t+j})$$

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$$A_t = \left(\frac{1}{1+r}\right)^T A_{t+T} + \sum_{j=0}^{T-1} \left(\frac{1}{1+r}\right)^j (c_{t+j} - y_{t+j})$$

What happens to the first term as $T \to \infty$?

If $r > 0$, as seems likely,

$$\lim_{T\to\infty} \left(\frac{1}{1+r}\right)^T = 0$$

But what rules out

$$\lim_{T\to\infty} A_{t+T} = \pm\infty$$

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If $\{A_t\}_{t=0}^{\infty}$ is bounded below by some finite limit on borrowing, then

$$\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^T A_{t+T} \geq 0$$

This no-Ponzi-game constraint will imply that

$$A_t = \lim_{T\to\infty}\left(\frac{1}{1+r}\right)^T A_{t+T} + \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j (c_{t+j} - y_{t+j})$$
$$\geq \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j (c_{t+j} - y_{t+j})$$

so that the consumer faces an infinite horizon present value budget constraint.

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Moreover, since $u' > 0$, consumer optimization will imply

$$\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^T A_{t+T} = 0$$

This transversality condition implies

$$\begin{aligned} A_t &= \lim_{T\to\infty}\left(\frac{1}{1+r}\right)^T A_{t+T} + \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j (c_{t+j} - y_{t+j}) \\ &= \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j (c_{t+j} - y_{t+j}) \end{aligned}$$

so that the consumer's infinite horizon present value budget constraint binds at the optimum.

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The period-by-period budget constraint, the no-Ponzi-game constraint, and the transversality condition therefore imply that the sequence $\{c_t\}_{t=0}^{\infty}$ that solves the dynamic optimization problem will satisfy

$$A_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j y_{t+j} = \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j c_{t+j} \qquad (16)$$

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$$v(A_t; t) = \max_{s_t} \; u(A_t + y_t - s_t) + \beta v((1+r)s_t; t+1)$$

FOC for s_t:

$$-u'(A_t + y_t - s_t) + \beta(1+r)v'((1+r)s_t; t+1) = 0$$

Envelope condition for A_t:

$$v'(A_t; t) = u'(A_t + y_t - s_t)$$

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$$-u'(A_t + y_t - s_t) + \beta(1+r)v'((1+r)s_t; t+1) = 0$$

$$v'(A_t; t) = u'(A_t + y_t - s_t)$$

Simplify using the constraints:

$$u'(c_t) = \beta(1+r)v'(A_{t+1}; t+1) \tag{17}$$

$$v'(A_t; t) = u'(c_t) \tag{18}$$

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$$u'(c_t) = \beta(1+r)v'(A_{t+1}; t+1) \tag{17}$$

$$v'(A_t; t) = u'(c_t) \tag{18}$$

Since (18) holds for all $t = 1, 2, 3, \ldots$, it implies

$$v'(A_{t+1}; t+1) = u'(c_{t+1}) \text{ for all } t = 0, 1, 2, \ldots$$

Substitute this expression into (17) to obtain

$$u'(c_t) = \beta(1+r)u'(c_{t+1}) \tag{19}$$

for all $t = 0, 1, 2, \ldots$.

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The *Euler equation*

$$u'(c_t) = \beta(1+r)u'(c_{t+1}) \tag{19}$$

is the discrete-time version of the Euler-Lagrange optimality condition from the calculus of variations.

It can be rewritten as the tangency condition

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r$$

equating the intertemporal marginal rate of substitution to the slope of the intertemporal budget constraint.

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Both in terms of math and economics, these manipulations are the same as those that use the static optimality conditions

$$U_a(c_a^*, c_b^*) - \lambda^* p_a = 0$$

$$U_b(c_a^*, c_b^*) - \lambda^* p_b = 0$$

to eliminate λ^* and arrive at the tangency condition

$$\frac{U_a(c_a^*, c_b^*)}{U_b(c_a^*, c_b^*)} = \frac{p_a}{p_b}$$

Just as we don't necessarily need a solution for λ^* to understand the solution to the static problem, we don't necessarily need a closed-form solution for the value function $v(A_t; t)$ to understand the solution to the dynamic problem.

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$$u'(c_t) = \beta(1+r)u'(c_{t+1}) \tag{19}$$

Now assume $\beta(1+r) = 1$. Then (19) implies

$$u'(c_t) = u'(c_{t+1}) \text{ for all } t = 0, 1, 2, \ldots$$

And since $u'' < 0$,

$$c_t = c_{t+j} \text{ for all } j = 0, 1, 2, \ldots$$

Saving Under Certainty

Substitute

$$c_t = c_{t+j} \text{ for all } j = 0, 1, 2, \ldots$$

into the binding present value budget constraint

$$A_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j y_{t+j} = \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j c_{t+j} \qquad (16)$$

to get

$$A_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j y_{t+j} = c_t \sum_{j=0}^{\infty} \beta^j \qquad (20)$$

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Remember from Euclid (300BC): If

$$X = \sum_{j=0}^{\infty} \beta^j = 1 + \beta + \beta^2 + \dots$$

then

$$\beta X = \beta + \beta^2 + \beta^3 + \dots$$

and hence

$$(1 - \beta) X = 1$$

or

$$\sum_{j=0}^{\infty} \beta^j = \frac{1}{1 - \beta}$$

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$$A_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j y_{t+j} = c_t \sum_{j=0}^{\infty} \beta^j \tag{20}$$

$$A_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j y_{t+j} = c_t \left(\frac{1}{1-\beta} \right)$$

$$c_t = (1-\beta) \left[A_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j y_{t+j} \right] \tag{21}$$

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$$c_t = (1-\beta)\left[A_t + \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j}\right] \qquad (21)$$

Equation (21) describes a version of the permanent income hypothesis: consumption depends on wealth, not just current income.

Since $\beta \approx 0.95$ on an annualized basis, (21) suggests that consumers spend a constant fraction (5 percent) of their wealth each period.