ECON 772001
MATH FOR ECONOMISTS

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Dynamic Programming Under Certainty

Given y_0, choose sequences $\{z_t\}_{t=0}^{\infty}$ and $\{y_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t F(y_t, z_t; t)$$

subject to

$$y_t + Q(y_t, z_t; t) \geq y_{t+1}$$

for all $t = 0, 1, 2, \ldots$ and

$$c \geq G(y_t, z_t; t)$$

for all $t = 0, 1, 2, \ldots$

Dynamic Programming

Define the value function recursively, using the Bellman equation

$$v(y_t; t) = \max_{z_t} F(y_t, z_t; t) + \beta v[y_t + Q(y_t, z_t; t); t + 1] \\ \text{subject to } c \geq G(y_t, z_t; t) \tag{5}$$

Dynamic Programming

$$
\begin{aligned}
v(y_t; t) = \max_{z_t} \; & F(y_t, z_t; t) + \beta v[y_t + Q(y_t, z_t; t); t+1] \\
& \text{subject to } c \geq G(y_t, z_t; t)
\end{aligned}
\tag{5}
$$

Applying the Kuhn-Tucker theorem to the static problem in (5), the FOC for z_t is

$$
\begin{aligned}
& F_2(y_t, z_t; t) + \beta v'[y_t + Q(y_t, z_t; t); t+1] Q_2(y_t, z_t; t) \\
& - \lambda_t G_2(y_t, z_t; t) = 0
\end{aligned}
\tag{6}
$$

Dynamic Programming

$$v(y_t;t) = \max_{z_t} \; F(y_t, z_t; t) + \beta v[y_t + Q(y_t, z_t; t); t+1] \quad \text{subject to } c \geq G(y_t, z_t; t) \tag{5}$$

Applying the Kuhn-Tucker theorem to the static problem in (5), the complementary slackness condition is

$$\lambda_t[c - G(y_t, z_t; t)] = 0 \tag{4}$$

Dynamic Programming

$$v(y_t;t) = \max_{z_t} \; F(y_t, z_t; t) + \beta v[y_t + Q(y_t, z_t; t); t+1] \quad (5)$$
$$\text{subject to } c \geq G(y_t, z_t; t)$$

Applying the envelope theorem to the static problem in (5):

$$\begin{aligned} v'(y_t;t) = {} & F_1(y_t, z_t; t) \\ & + \beta v'[y_t + Q(y_t, z_t; t); t+1][1 + Q_1(y_t, z_t; t)] \\ & - \lambda_t G_1(y_t, z_t; t) \end{aligned} \quad (7)$$

Dynamic Programming

Together with the binding constraint

$$y_{t+1} = y_t + Q(y_t, z_t; t) \tag{3}$$

the FOC (6) for z_t, the complementary slackness condition (4), and the *envelope condition* (7) form a system of 4 equations in 4 unknowns: y_t, z_t, λ_t, and $v(\cdot; t)$.

Moreover, (6) and (7) coincide with (1) and (2), with

$$\mu_t = W(y_t; t) = v'(y_t; t) = \text{"MU of } y_t\text{"}$$

And for μ_{t+1} as a function of y_{t+1}:

$$\mu_{t+1} = W(y_{t+1}; t+1) = v'(y_{t+1}; t+1) = \text{"MU of } y_{t+1}\text{"}$$

Dynamic Programming

Together with the binding constraint

$$y_{t+1} = y_t + Q(y_t, z_t; t) \tag{3}$$

the FOC (6) for z_t, the complementary slackness condition (4), and the *envelope condition* (7) form a system of 4 equations in 4 unknowns: y_t, z_t, λ_t, and $v(\cdot; t)$.

But is it possible to solve a system of equations that involves the unknown function $v(\cdot; t)$ instead of the unknown variable μ_t?

Dynamic Programming

But is it possible to solve a system of equations that involves the unknown function $v(\cdot;t)$ instead of the unknown variable μ_t?

Two examples will illustrate some possibilities:

Ex 1: Optimal Growth – a rare case where we can solve explicitly for the value function.

Ex 2: Saving Under Certainty – a more typical case where we can use dynamic program to characterize the solution to the dynamic optimization problem without explicitly solving for the value function.

Optimal Growth

A special case of the Ramsey model in discrete time, with:

Log utility

Cobb-Douglas production

Complete depreciation, $\delta = 1$

Optimal Growth

k_t = state variable

c_t = control variable

With $\delta = 1$:

$$k_{t+1} = k_t + k_t^\alpha - \delta k_t - c_t = k_t^\alpha - c_t$$

$$k_t^\alpha \geq c_t + k_{t+1}$$

for all $t = 0, 1, 2, \ldots$.

Optimal Growth

$$k_t^\alpha \geq c_t + k_{t+1}$$

$k_t^\alpha =$ output (real GDP)

$c_t =$ consumption

$k_{t+1} =$ investment/savings

Optimal Growth

The social planner's problem: given k_0, choose sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$k_t^{\alpha} \geq c_t + k_{t+1}$$

for all $t = 0, 1, 2, \ldots$.

Optimal Growth

Given k_0, choose sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$k_t^{\alpha} \geq c_t + k_{t+1}$$

for all $t = 0, 1, 2, \ldots$.

The Bellman equation:

$$v(k_t; t) = \max_{c_t} \ \ln(c_t) + \beta v(k_t^{\alpha} - c_t; t+1)$$

Optimal Growth

The Bellman equation:

$$v(k_t; t) = \max_{c_t} \ \ln(c_t) + \beta v(k_t^{\alpha} - c_t; t+1)$$

Even with $\delta < 1$, it will be true for the Ramsey model that

$$v(k_t; t) = v(k_t)$$

The capital stock k_t summarizes completely the state of the world as it appears to the social planner.

Optimal Growth

The Bellman equation:

$$v(k_t; t) = \max_{c_t} \ \ln(c_t) + \beta v(k_t^\alpha - c_t; t+1)$$

But with $\delta = 1$, it will also be true that

$$v(k_t; t) = v(k_t) = E + F \ln(k_t)$$

where E and F are *undetermined coefficients*.

Optimal Growth

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$$v(k_t; t) = v(k_t) = E + F \ln(k_t)$$

where E and F are *undetermined coefficients*.

Note that if we use the guess-and-verify method, ideally we should establish first that there is a unique function $v(k_t; t)$ that satisfies the Bellman equation.

Optimal Growth

Using the guess, the Bellman equation:

$$v(k_t; t) = \max_{c_t} \ln(c_t) + \beta v(k_t^\alpha - c_t; t+1)$$

becomes

$$E + F\ln(k_t) = \max_{c_t} \ \ln(c_t) + \beta E + \beta F \ln(k_t^\alpha - c_t) \quad (8)$$

Optimal Growth

$$E + F\ln(k_t) = \max_{c_t} \ \ln(c_t) + \beta E + \beta F \ln(k_t^{\alpha} - c_t) \quad (8)$$

FOC for c_t:

$$\frac{1}{c_t} - \frac{\beta F}{k_t^{\alpha} - c_t} = 0 \quad (9)$$

Envelope condition for k_t:

$$\frac{F}{k_t} = \frac{\alpha\beta F k_t^{\alpha-1}}{k_t^{\alpha} - c_t} \quad (10)$$

Optimal Growth

$$E + F\ln(k_t) = \max_{c_t}\ \ln(c_t) + \beta E + \beta F \ln(k_t^\alpha - c_t) \tag{8}$$

$$\frac{1}{c_t} - \frac{\beta F}{k_t^\alpha - c_t} = 0 \tag{9}$$

$$\frac{F}{k_t} = \frac{\alpha\beta F k_t^{\alpha-1}}{k_t^\alpha - c_t} \tag{10}$$

Together with the binding constraint

$$k_{t+1} = k_t^\alpha - c_t$$

(8)-(10) form a system of 4 equations in 4 unknowns: c_t, k_t, E, and F.

Optimal Growth

$$\frac{1}{c_t} - \frac{\beta F}{k_t^\alpha - c_t} = 0 \tag{9}$$

$$k_t^\alpha - c_t = \beta F c_t$$

$$c_t = \left(\frac{1}{1 + \beta F}\right) k_t^\alpha \tag{11}$$

Optimal Growth

$$\frac{F}{k_t} = \frac{\alpha\beta F k_t^{\alpha-1}}{k_t^{\alpha} - c_t} \tag{10}$$

$$F k_t^{\alpha} - F c_t = \alpha\beta F k_t^{\alpha}$$

Using (11):

$$F k_t^{\alpha} - F\left(\frac{1}{1+\beta F}\right) k_t^{\alpha} = \alpha\beta F k_t^{\alpha}$$

Optimal Growth

$$Fk_t^\alpha - F\left(\frac{1}{1+\beta F}\right)k_t^\alpha = \alpha\beta Fk_t^\alpha$$

$$1 - \frac{1}{1+\beta F} = \alpha\beta$$

$$\frac{1}{1+\beta F} = 1 - \alpha\beta \tag{12}$$

Optimal Growth

$$\frac{1}{1+\beta F} = 1 - \alpha\beta \tag{12}$$

$$1 + \beta F = \frac{1}{1-\alpha\beta}$$

$$\beta F = \frac{1}{1-\alpha\beta} - \frac{1-\alpha\beta}{1-\alpha\beta} = \frac{\alpha\beta}{1-\alpha\beta}$$

Confirming that F is a constant:

$$F = \frac{\alpha}{1-\alpha\beta} \tag{13}$$

Optimal Growth

Combine (11) and (12)

$$c_t = \left(\frac{1}{1+\beta F}\right) k_t^{\alpha} \tag{11}$$

$$\frac{1}{1+\beta F} = 1 - \alpha\beta \tag{12}$$

to get

$$c_t = (1-\alpha\beta) k_t^{\alpha} \tag{14}$$

In this special case, it is optimal to consume a fixed fraction of output.

Optimal Growth

Substitute

$$c_t = (1 - \alpha\beta)k_t^\alpha \tag{14}$$

into the binding constraint

$$k_{t+1} = k_t^\alpha - c_t$$

to get

$$k_{t+1} = \alpha\beta k_t^\alpha \tag{15}$$

In this special case of the Ramsey model, Solow's assumption of a fixed savings rate holds.

Optimal Growth

$$c_t = (1 - \alpha\beta)k_t^\alpha \tag{14}$$

$$k_{t+1} = \alpha\beta k_t^\alpha \tag{15}$$

Given k_0, (14) and (15) imply values for c_0 and k_1. Given k_1, (14) and (15) imply values for c_1 and k_2.

The optimal sequences $\{c_t\}_{t=0}^\infty$ and $\{k_t\}_{t=1}^\infty$ can be computed knowing the value $c_0 = (1 - \alpha\beta)k_0^\alpha$ that puts the economy on the saddle path.

Optimal Growth

To see the qualitative properties of the solution, use

$$k_{t+1} = \alpha\beta k_t^{\alpha} \tag{15}$$

$$\ln(k_{t+1}) = \ln(\alpha\beta) + \alpha \ln(k_t)$$

$$\ln(k_{t+1}) = (1-\alpha)\left[\frac{\ln(\alpha\beta)}{1-\alpha}\right] + \alpha \ln(k_t)$$

$$\ln(k_{t+1}) - \frac{\ln(\alpha\beta)}{1-\alpha} = \alpha\left[\ln(k_t) - \frac{\ln(\alpha\beta)}{1-\alpha}\right]$$

Optimal Growth

$$\ln(k_{t+1}) - \frac{\ln(\alpha\beta)}{1-\alpha} = \alpha\left[\ln(k_t) - \frac{\ln(\alpha\beta)}{1-\alpha}\right]$$

or

$$z_{t+1} = \alpha z_t$$

where

$$z_t = \ln(k_t) - \frac{\ln(\alpha\beta)}{1-\alpha}$$

Optimal Growth

$$z_{t+1} = \alpha z_t$$

where

$$z_t = \ln(k_t) - \frac{\ln(\alpha\beta)}{1-\alpha}$$

Since $0 < \alpha < 1$, starting from any value of z_0, corresponding to any given k_0:

$$\lim_{t\to\infty} z_t = 0$$

$$\lim_{t\to\infty} \ln(k_t) = \frac{\ln(\alpha\beta)}{1-\alpha}$$

Optimal Growth

$$\lim_{t\to\infty} \ln(k_t) = \frac{\ln(\alpha\beta)}{1-\alpha} = \ln[(\alpha\beta)^{1/(1-\alpha)}]$$

$$\lim_{t\to\infty} k_t = (\alpha\beta)^{1/(1-\alpha)} = k^*$$

$$\lim_{t\to\infty} c_t = (1-\alpha\beta)(k^*)^{\alpha}$$

Recall that in the more general case where $\delta \leq 1$,

$$k^* = \left[\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right]^{1/(\alpha-1)}$$

Set $\delta = 1$ to recover this special case considered here.

Optimal Growth

Finally, to solve for E, return to

$$E + F\ln(k_t) = \max_{c_t}\ \ln(c_t) + \beta E + \beta F\ln(k_t^\alpha - c_t) \tag{8}$$

and substitute

$$c_t = (1-\alpha\beta)k_t^\alpha \tag{14}$$

into the right-hand side:

$$E + F\ln(k_t) = \ln[(1-\alpha\beta)k_t^\alpha] + \beta E + \beta F\ln(\alpha\beta k_t^\alpha)$$

Optimal Growth

$$E + F\ln(k_t) = \ln[(1-\alpha\beta)k_t^{\alpha}] + \beta E + \beta F\ln(\alpha\beta k_t^{\alpha})$$

$$E+F\ln(k_t) = \ln(1-\alpha\beta)+\alpha\ln(k_t)+\beta E+\beta F\ln(\alpha\beta)+\alpha\beta F\ln(k_t)$$

Since (13) implies

$$\begin{aligned}\alpha + \alpha\beta F &= \alpha + \alpha\beta\left(\frac{\alpha}{1-\alpha\beta}\right) \\ &= \frac{\alpha(1-\alpha\beta)+\alpha^2\beta}{1-\alpha\beta} = \frac{\alpha}{1-\alpha\beta} = F\end{aligned}$$

the terms involving $\ln(k_t)$ cancel.

Optimal Growth

$$E+F\ln(k_t) = \ln(1-\alpha\beta)+\alpha\ln(k_t)+\beta E+\beta F\ln(\alpha\beta)+\alpha\beta F\ln(k_t)$$

$$E = \ln(1-\alpha\beta) + \beta E + \beta F\ln(\alpha\beta)$$

$$E = \frac{1}{1-\beta}\left[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)\right]$$

Confirming that E is constant as well.