# ECON 772001
# MATH FOR ECONOMISTS

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# Dynamic Programming Under Certainty

Given $y_0$, choose sequences $\{z_t\}_{t=0}^{\infty}$ and $\{y_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t F(y_t, z_t; t)$$

subject to

$$y_t + Q(y_t, z_t; t) \geq y_{t+1}$$

for all $t = 0, 1, 2, \ldots$ and

$$c \geq G(y_t, z_t; t)$$

for all $t = 0, 1, 2, \ldots$

# Method of Lagrange Multipliers

$$L(\{z_t\}_{t=0}^{\infty}, \{y_t\}_{t=1}^{\infty}, \{\tilde{\mu}_t\}_{t=1}^{\infty}, \{\tilde{\lambda}_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t F(y_t, z_t; t)$$

$$+ \sum_{t=0}^{\infty} \tilde{\mu}_{t+1}[y_t + Q(y_t, z_t; t) - y_{t+1}] + \sum_{t=0}^{\infty} \tilde{\lambda}_t [c - G(y_t, z_t; t)]$$

Note that the multipliers are in "present value" form. To convert to "current values," use the change of variables

$$\tilde{\mu}_{t+1} = \beta^{t+1} \mu_{t+1}$$

and

$$\tilde{\lambda}_t = \beta^t \lambda_t$$

# Method of Lagrange Multipliers

$$L(\{z_t\}_{t=0}^{\infty}, \{y_t\}_{t=1}^{\infty}, \{\tilde{\mu}_t\}_{t=1}^{\infty}, \{\tilde{\lambda}_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t F(y_t, z_t; t)$$

$$+ \sum_{t=0}^{\infty} \tilde{\mu}_{t+1}[y_t + Q(y_t, z_t; t) - y_{t+1}] + \sum_{t=0}^{\infty} \tilde{\lambda}_t[c - G(y_t, z_t; t)]$$

$$L(\{z_t\}_{t=0}^{\infty}, \{y_t\}_{t=1}^{\infty}, \{\mu_t\}_{t=1}^{\infty}, \{\lambda_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t F(y_t, z_t; t)$$

$$+ \sum_{t=0}^{\infty} \beta^{t+1} \mu_{t+1}[y_t + Q(y_t, z_t; t) - y_{t+1}]$$

$$+ \sum_{t=0}^{\infty} \beta^t \lambda_t[c - G(y_t, z_t; t)]$$

# Method of Lagrange Multipliers

$$L = \sum_{t=0}^{\infty} \beta^t F(y_t, z_t; t) + \sum_{t=0}^{\infty} \beta^{t+1} \mu_{t+1} [y_t + Q(y_t, z_t; t) - y_{t+1}]$$
$$+ \sum_{t=0}^{\infty} \beta^t \lambda_t [c - G(y_t, z_t; t)]$$

Fix $t = 0, 1, 2, \ldots$. FOC for $z_t$:

$$\beta^t F_2(y_t, z_t; t) + \beta^{t+1} \mu_{t+1} Q_2(y_t, z_t; t) - \beta^t \lambda_t G_2(y_t, z_t; t) = 0$$

# Method of Lagrange Multipliers

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$$+ \sum_{t=0}^{\infty} \beta^t \lambda_t [c - G(y_t, z_t; t)]$$

Fix $t = 1, 2, 3, \ldots$. FOC for $y_t$:

$$\beta^t F_1(y_t, z_t; t) + \beta^{t+1} \mu_{t+1} [1 + Q_1(y_t, z_t; t)]$$
$$- \beta^t \mu_t - \beta^t \lambda_t G_1(y_t, z_t; t) = 0$$

# Method of Lagrange Multipliers

Suppose it is possible to solve for $\mu_t$ as a function of $y_t$ (and perhaps other variables as well):

$$\mu_t = W(y_t; t) = \text{“MU of } y_t\text{”}$$

And for $\mu_{t+1}$ as a function of $y_{t+1}$:

$$\begin{aligned}\mu_{t+1} &= W(y_{t+1}; t+1) \\ &= W[y_t + Q(y_t, z_t; t); t+1] = \text{“MU of } y_{t+1}\text{”}\end{aligned}$$

# Method of Lagrange Multipliers

FOC for $z_t$:

$$\beta^t F_2(y_t, z_t; t) + \beta^{t+1}\mu_{t+1} Q_2(y_t, z_t; t) - \beta^t \lambda_t G_2(y_t, z_t; t) = 0$$

$$\begin{aligned} &F_2(y_t, z_t; t) + \beta W[y_t + Q(y_t, z_t; t); t+1] Q_2(y_t, z_t; t) \\ &- \lambda_t G_2(y_t, z_t; t) = 0 \end{aligned} \tag{1}$$

for all $t = 0, 1, 2, \ldots$

# Method of Lagrange Multipliers

FOC for $y_t$:

$$\beta^t F_1(y_t, z_t; t) + \beta^{t+1}\mu_{t+1}[1 + Q_1(y_t, z_t; t)]$$
$$- \beta^t \mu_t - \beta^t \lambda_t G_1(y_t, z_t; t) = 0$$

$$\begin{aligned} W(y_t; t) = \; & F_1(y_t, z_t; t) \\ & + \beta W[y_t + Q(y_t, z_t; t); t+1][1 + Q_1(y_t, z_t; t)] \\ & - \lambda_t G_1(y_t, z_t; t) \end{aligned} \tag{2}$$

for all $t = 1, 2, 3, \ldots$

# Method of Lagrange Multipliers

Together with the binding constraint

$$y_{t+1} = y_t + Q(y_t, z_t; t) \tag{3}$$

for all $t = 0, 1, 2, \ldots$, and the complementary slackness condition

$$\lambda_t[c - G(y_t, z_t; t)] = 0 \tag{4}$$

for all $t = 0, 1, 2, \ldots$, (1) and (2) form a system of 4 equations in 4 unknowns: $y_t$, $z_t$, $\lambda_t$, and $W(\cdot; t)$.

# Dynamic Programming

At $t = 0$, take $y_0$ as a parameter, and define the (maximum) value function

$$v(y_0; t = 0) = \max_{\{z_t\}_{t=0}^{\infty}, \{y_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(y_t, z_t; t)$$

subject to

$$y_t + Q(y_t, z_t; t) \geq y_{t+1} \text{ for all } t = 0, 1, 2, \ldots$$

$$c \geq G(y_t, z_t; t) \text{ for all } t = 0, 1, 2, \ldots$$

# Dynamic Programming

More generally, for any $t = 0, 1, 2, \ldots$, take $y_t$ as a parameter, and define the (maximum) value function

$$v(y_t; t) = \max_{\{z_{t+j}\}_{j=0}^{\infty}, \{y_{t+j}\}_{j=1}^{\infty}} \sum_{j=0}^{\infty} \beta^j F(y_{t+j}, z_{t+j}; t+j)$$

subject to

$$y_{t+j} + Q(y_{t+j}, z_{t+j}; t+j) \geq y_{t+j+1} \text{ for all } j = 0, 1, 2, \ldots$$

$$c \geq G(y_{t+j}, z_{t+j}; t+j) \text{ for all } j = 0, 1, 2, \ldots$$

# Dynamic Programming

Separate out the time-$t$ components:

$$v(y_t;t) = \max_{z_t, y_{t+1}} \left[ F(y_t, z_t; t) + \max_{\{z_{t+j}\}_{j=1}^{\infty}, \{y_{t+j}\}_{j=2}^{\infty}} \sum_{j=1}^{\infty} \beta^j F(y_{t+j}, z_{t+j}; t+j) \right]$$

subject to

$$y_t + Q(y_t, z_t; t) \geq y_{t+1}, \;\; c \geq G(y_t, z_t; t)$$

$$y_{t+j} + Q(y_{t+j}, z_{t+j}; t+j) \geq y_{t+j+1} \text{ for all } j = 1, 2, 3, \ldots$$

$$c \geq G(y_{t+j}, z_{t+j}; t+j) \text{ for all } j = 1, 2, 3, \ldots$$

# Dynamic Programming

$$v(y_t;t) = \max_{z_t, y_{t+1}} \left[ F(y_t, z_t; t) + \max_{\{z_{t+j}\}_{j=1}^{\infty}, \{y_{t+j}\}_{j=2}^{\infty}} \sum_{j=1}^{\infty} \beta^j F(y_{t+j}, z_{t+j}; t+j) \right]$$

subject to

$$y_t + Q(y_t, z_t; t) \geq y_{t+1}, \;\; c \geq G(y_t, z_t; t)$$

$$y_{t+j} + Q(y_{t+j}, z_{t+j}; t+j) \geq y_{t+j+1} \text{ for all } j = 1, 2, 3 \ldots$$

$$c \geq G(y_{t+j}, z_{t+j}; t+j) \text{ for all } j = 1, 2, 3, \ldots$$

Relable the time indices:

$$v(y_t;t) = \max_{z_t, y_{t+1}} \left[ F(y_t, z_t; t) + \beta \max_{\{z_{t+1+j}\}_{j=0}^{\infty}, \{y_{t+1+j}\}_{j=1}^{\infty}} \sum_{j=0}^{\infty} \beta^j F(y_{t+1+j}, z_{t+1+j}; t+1+j) \right]$$

subject to

$$y_t + Q(y_t, z_t; t) \geq y_{t+1}, \;\; c \geq G(y_t, z_t; t)$$

$$y_{t+1+j} + Q(y_{t+1+j}, z_{t+1+j}; t+1+j) \geq y_{t+1+j+1} \text{ for all } j = 0, 1, 2, \ldots$$

$$c \geq G(y_{t+1+j}, z_{t+1+j}; t+1+j) \text{ for all } j = 0, 1, 2, \ldots$$

# Dynamic Programming

Relable the time indices:

$$v(y_t; t) = \max_{z_t, y_{t+1}} \left[ F(y_t, z_t; t) + \beta \max_{\{z_{t+1+j}\}_{j=0}^{\infty}, \{y_{t+1+j}\}_{j=1}^{\infty}} \sum_{j=0}^{\infty} \beta^j F(y_{t+1+j}, z_{t+1+j}; t+1+j) \right]$$

subject to

$$y_t + Q(y_t, z_t; t) \geq y_{t+1}, \;\; c \geq G(y_t, z_t; t)$$

$$y_{t+1+j} + Q(y_{t+1+j}, z_{t+1+j}; t+1+j) \geq y_{t+1+j+1} \text{ for all } j = 0, 1, 2, \ldots$$

$$c \geq G(y_{t+1+j}, z_{t+1+j}; t+1+j) \text{ for all } j = 0, 1, 2, \ldots$$

The terms in red define $v(y_{t+1}; t+1)$!

# Dynamic Programming

$$v(y_t;t) = \max_{z_t, y_{t+1}} \left[ F(y_t, z_t; t) + \beta \max_{\{z_{t+1+j}\}_{j=0}^{\infty}, \{y_{t+1+j}\}_{j=1}^{\infty}} \sum_{j=0}^{\infty} \beta^j F(y_{t+1+j}, z_{t+1+j}; t+1+j) \right]$$

subject to

$$y_t + Q(y_t, z_t; t) \geq y_{t+1}, \;\; c \geq G(y_t, z_t; t)$$

$$y_{t+1+j} + Q(y_{t+1+j}, z_{t+1+j}; t+1+j) \geq y_{t+1+j+1} \text{ for all } j = 0, 1, 2, \ldots$$

$$c \geq G(y_{t+1+j}, z_{t+1+j}; t+1+j) \text{ for all } j = 0, 1, 2, \ldots$$

$$v(y_t;t) = \max_{z_t, y_{t+1}} \left[ F(y_t, z_t; t) + \beta v(y_{t+1}; t+1) \right]$$

subject to

$$y_t + Q(y_t, z_t; t) \geq y_{t+1}, \;\; c \geq G(y_t, z_t; t)$$

# Dynamic Programming

$$v(y_t;t) = \max_{z_t, y_{t+1}} [F(y_t, z_t; t) + \beta v(y_{t+1}; t+1)]$$

subject to

$$y_t + Q(y_t, z_t; t) \geq y_{t+1}, \; c \geq G(y_t, z_t; t)$$

$$v(y_t;t) = \max_{z_t} \; F(y_t, z_t; t) + \beta v[y_t + Q(y_t, z_t; t); t+1] \quad \text{subject to } c \geq G(y_t, z_t; t) \tag{5}$$

In practice, the *Bellman equation* (5) is the starting point for the dynamic programming approach.

# Dynamic Programming

$$v(y_t;t) = \max_{z_t} \; F(y_t, z_t; t) + \beta v[y_t + Q(y_t, z_t; t); t+1] \qquad (5)$$
$$\text{subject to } c \geq G(y_t, z_t; t)$$

The Bellman equation is a recursive definition of the value function $v(y_t;t)$.

The optimization problem on the right-hand side takes the form of a static optimization problem with a single choice variable and a single constraint.

# Dynamic Programming

$$v(y_t; t) = \max_{z_t} \; F(y_t, z_t; t) + \beta v[y_t + Q(y_t, z_t; t); t+1] \quad \text{subject to } c \geq G(y_t, z_t; t) \tag{5}$$

Applying the Kuhn-Tucker theorem to the static problem in (5), the FOC for $z_t$ is

$$F_2(y_t, z_t; t) + \beta v'[y_t + Q(y_t, z_t; t); t+1] Q_2(y_t, z_t; t) - \lambda_t G_2(y_t, z_t; t) = 0 \tag{6}$$

# Dynamic Programming

$$v(y_t;t) = \max_{z_t} \; F(y_t, z_t; t) + \beta v[y_t + Q(y_t, z_t; t); t+1] \qquad (5)$$
$$\text{subject to } c \geq G(y_t, z_t; t)$$

Applying the Kuhn-Tucker theorem to the static problem in (5), the complementary slackness condition is

$$\lambda_t[c - G(y_t, z_t; t)] = 0 \qquad (4)$$

# Dynamic Programming

$$v(y_t;t) = \max_{z_t} \; F(y_t,z_t;t) + \beta v[y_t + Q(y_t,z_t;t);t+1] \quad \text{subject to } c \geq G(y_t,z_t;t) \tag{5}$$

Applying the envelope theorem to the static problem in (5):

$$\begin{aligned} v'(y_t;t) = {} & F_1(y_t,z_t;t) \\ & + \beta v'[y_t + Q(y_t,z_t;t);t+1][1 + Q_1(y_t,z_t;t)] \\ & - \lambda_t G_1(y_t,z_t;t) \end{aligned} \tag{7}$$

# Dynamic Programming

Together with the binding constraint

$$y_{t+1} = y_t + Q(y_t, z_t; t) \tag{3}$$

the FOC (6) for $z_t$, the complementary slackness condition (4), and the *envelope condition* (7) form a system of 4 equations in 4 unknowns: $y_t$, $z_t$, $\lambda_t$, and $v(\cdot; t)$.

Moreover, (6) and (7) coincide with (1) and (2), with

$$\mu_t = W(y_t; t) = v'(y_t; t) = \text{ “MU of } y_t\text{”}$$

And for $\mu_{t+1}$ as a function of $y_{t+1}$:

$$\mu_{t+1} = W(y_{t+1}; t+1) = v'(y_{t+1}; t+1) = \text{ “MU of } y_{t+1}\text{”}$$