ECON 772001
MATH FOR ECONOMISTS

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November 3, 2020

The Ramsey Model

Given $k(0)$, choose functions $c(t)$, $t \in [0, \infty)$, and $k(t)$, $t \in (0, \infty)$, to maximize

$$\int_0^\infty e^{-\rho t} \ln(c(t)) dt$$

subject to

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$.

The Ramsey Model

For this problem, we can define the present value Hamiltonian via

$$H(k(t), \pi(t), t) = \max_{c(t)} e^{-\rho t} \ln(c(t)) + \pi(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

or the current value Hamiltonian via

$$\bar{H}(k(t), \theta(t)) = \max_{c(t)} \ln(c(t)) + \theta(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

The Ramsey Model

Comparing

$$H(k(t), \pi(t), t) = \max_{c(t)} \; e^{-\rho t} \ln(c(t)) + \pi(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

$$\bar{H}(k(t), \theta(t)) = \max_{c(t)} \; \ln(c(t)) + \theta(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

reveals that

$$\pi(t) = e^{-\rho t}\theta(t)$$

and hence

$$\dot{\pi}(t) = -\rho e^{-\rho t}\theta(t) + e^{-\rho t}\dot{\theta}(t)$$

The Ramsey Model

$$H(k(t),\pi(t),t) = \max_{c(t)}\ e^{-\rho t}\ln(c(t)) + \pi(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

$$\bar{H}(k(t),\theta(t)) = \max_{c(t)}\ \ln(c(t)) + \theta(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

FOC for $c(t)$ based on PV

$$\frac{e^{-\rho t}}{c(t)} - \pi(t) = 0$$

can be rewritten as the FOC based on CV:

$$\frac{e^{-\rho t}}{c(t)} - e^{-\rho t}\theta(t) = 0 \text{ or } \frac{1}{c(t)} - \theta(t) = 0$$

The Ramsey Model

$$H(k(t), \pi(t), t) = \max_{c(t)} \; e^{-\rho t} \ln(c(t)) + \pi(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

$$\bar{H}(k(t), \theta(t)) = \max_{c(t)} \; \ln(c(t)) + \theta(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

Differential equation for $\pi(t)$ based on PV

$$\dot{\pi}(t) = -H_k(k(t), \pi(t), t) = -\pi(t)[\alpha k(t)^{\alpha-1} - \delta]$$

can be rewritten as the differential equation for $\theta(t)$ based on CV:

$$-\rho e^{-\rho t}\theta(t) + e^{-\rho t}\dot{\theta}(t) = -e^{-\rho t}\theta(t)[\alpha k(t)^{\alpha-1} - \delta]$$

$$\dot{\theta}(t) = \rho\theta(t) - \theta(t)[\alpha k(t)^{\alpha-1} - \delta] = \rho\theta(t) - \bar{H}_k(k(t), \theta(t))$$

The Ramsey Model

$$H(k(t), \pi(t), t) = \max_{c(t)} e^{-\rho t} \ln(c(t)) + \pi(t)[k(t)^\alpha - \delta k(t) - c(t)]$$

$$\bar{H}(k(t), \theta(t)) = \max_{c(t)} \ln(c(t)) + \theta(t)[k(t)^\alpha - \delta k(t) - c(t)]$$

Differential equation for $k(t)$ based on PV

$$\dot{k}(t) = H_\pi(k(t), \pi(t), t) = k(t)^\alpha - \delta k(t) - c(t)$$

is of course that same as the differential equation for $k(t)$ based on PV

$$\dot{k}(t) = \bar{H}_\theta(k(t), \pi(t), t) = k(t)^\alpha - \delta k(t) - c(t)$$

The Ramsey Model

Similar observations apply in the discrete time case.

See the notes on the maximum principle for details.

The main thing is to remember to include the extra term $\rho\theta(t)$ in the differential equation for $\theta(t)$ when using the current value Hamiltonian.

Dynamic Programming

Dynamic Programming

- Stochastic Dynamic Programming
  - A Dynamic Stochastic Problem
  - The Dynamic Programming Formulation
- Example 3: Saving with Multiple Random Returns
- The Kuhn-Tucker Theorem with Uncertainty

Dynamic Programming

References:

Dixit, Ch 11

Acemoglu, Chs 6 and 16

Stokey, Lucas, Prescott *Recursive Methods in Economic Dynamics*

Hans Joseph Pesch and Michael Plail. "The Cold War and the Maximum Principle of Optimal Control." *Documenta Mathematica* (2012), pp.331-343.

Dynamic Programming Under Certainty

Discrete time, infinite horizon: $t = 0, 1, 2, \ldots$

No uncertainty

y_t = stock, or state, variable at the beginning of t

z_t = flow, or control, variable during t

Dynamic Programming Under Certainty

Objective function:

$$\sum_{t=0}^{\infty} \beta^t F(y_t, z_t; t)$$

Constraints:

$$y_0 \text{ given}$$

$$y_t + Q(y_t, z_t; t) \geq y_{t+1} \text{ for all } t = 0, 1, 2, \ldots$$

$$c \geq G(y_t, z_t; t) \text{ for all } t = 0, 1, 2, \ldots$$