ECON 772001
MATH FOR ECONOMISTS

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A Continuous Time Problem

The problem: Given $y(0)$, choose functions $z(t), t \in [0, T]$, and $y(t), t \in (0, T]$, to maximize

$$\int_0^T e^{-\rho t} F(y(t), z(t); t) dt$$

subject to the constraints

$$Q(y(t), z(t); t) \geq \dot{y}(t) \text{ for all } t \in [0, T]$$

$$c \geq G(y(t), z(t); t) \text{ for all } t \in [0, T]$$

$$y(T) \geq y^*$$

The Kuhn-Tucker Formulation

$$\begin{aligned}L(z(t), y(t), \pi(t), \lambda(t), \phi) = &\int_0^T e^{-\rho t} F(y(t), z(t); t)dt \\ &+ \int_0^T \pi(t)[Q(y(t), z(t); t) - \dot{y}(t)]dt \\ &+ \int_0^T \lambda(t)[c - G(y(t), z(t); t)]dt \\ &+ \phi[y(T) - y^*]\end{aligned}$$

Use integration by parts to substitute out for the term involving $\pi(t)\dot{y}(t)$ in the "present value" Lagrangian.

The Kuhn-Tucker Formulation

$$\frac{d\pi(t)y(t)}{dt} = \dot{\pi}(t)y(t) + \pi(t)\dot{y}(t)$$

$$\int_0^T \left[\frac{d\pi(t)y(t)}{dt}\right] dt = \int_0^T \dot{\pi}(t)y(t)dt + \int_0^T \pi(t)\dot{y}(t)dt$$

$$\pi(T)y(T) - \pi(0)y(0) = \int_0^T \dot{\pi}(t)y(t)dt + \int_0^T \pi(t)\dot{y}(t)dt$$

$$-\int_0^T \pi(t)\dot{y}(t)dt = \int_0^T \dot{\pi}(t)y(t)dt + \pi(0)y(0) - \pi(T)y(T)$$

The Kuhn-Tucker Formulation

$$
\begin{aligned}
L(z(t), y(t), \pi(t), \lambda(t), \phi) = & \int_0^T e^{-\rho t} F(y(t), z(t); t) dt \\
& + \int_0^T \pi(t) Q(y(t), z(t); t) dt \\
& + \int_0^T \dot{\pi}(t) y(t) dt \\
& + \pi(0) y(0) - \pi(T) y(T) \\
& + \int_0^T \lambda(t) [c - G(y(t), z(t); t)] dt \\
& + \phi [y(T) - y^*]
\end{aligned}
$$

The Kuhn-Tucker Formulation

$$L = \int_0^T e^{-\rho t} F(y(t), z(t); t) dt + \int_0^T \pi(t) Q(y(t), z(t); t) dt + \int_0^T \dot{\pi}(t) y(t) dt$$
$$+ \pi(0) y(0) - \pi(T) y(T) + \int_0^T \lambda(t) [c - G(y(t), z(t); t)] dt + \phi[y(T) - y^*]$$

Fix $t \in [0, T]$ "FOC" for $z(t)$:

$$e^{-\rho t} F_z(y(t), z(t); t) + \pi(t) Q_z(y(t), z(t); t) - \lambda(t) G_z(y(t), z(t); t) = 0 \quad (13)$$

for all $t \in [0, T]$.

The Kuhn-Tucker Formulation

$$L = \int_0^T e^{-\rho t} F(y(t), z(t); t) dt + \int_0^T \pi(t) Q(y(t), z(t); t) dt + \int_0^T \dot{\pi}(t) y(t) dt$$
$$+ \pi(0) y(0) - \pi(T) y(T) + \int_0^T \lambda(t) [c - G(y(t), z(t); t)] dt + \phi[y(T) - y^*]$$

Fix $t \in (0, T)$ "FOC" for $y(t)$:

$$e^{-\rho t} F_y(y(t), z(t); t) + \pi(t) Q_y(y(t), z(t); t) + \dot{\pi}(t)$$
$$- \lambda(t) G_y(y(t), z(t); t) = 0$$

for all $t \in (0, T)$.

The Kuhn-Tucker Formulation

$$e^{-\rho t}F_y(y(t),z(t);t)+\pi(t)Q_y(y(t),z(t);t)+\dot{\pi}(t) \\ -\lambda(t)G_y(y(t),z(t);t)=0$$

for all $t \in (0, T)$.

If all functions of t are continuously differentiable:

$$\dot{\pi}(t)=-[e^{-\rho t}F_y(y(t),z(t);t)+\pi(t)Q_y(y(t),z(t);t) \\ -\lambda(t)G_y(y(t),z(t);t)] \tag{14}$$

for all $t \in [0, T]$.

The Kuhn-Tucker Formulation

$$L = \int_0^T e^{-\rho t} F(y(t), z(t); t) dt + \int_0^T \pi(t) Q(y(t), z(t); t) dt + \int_0^T \dot{\pi}(t) y(t) dt$$
$$+ \pi(0) y(0) - \pi(T) y(T) + \int_0^T \lambda(t)[c - G(y(t), z(t); t)] dt + \phi[y(T) - y^*]$$

"FOC" for $y(T)$:

$$e^{-\rho T} F_y(y(T), z(T); T) + \pi(T) Q_y(y(T), z(T); T) + \dot{\pi}(T)$$
$$- \pi(T) - \lambda(T) G_y(y(T), z(T); T) + \phi = 0$$

$$\phi = \pi(T)$$

The Kuhn-Tucker Formulation

Together with the binding constraint

$$\dot{y}(t) = Q(y(t), z(t); t) \tag{15}$$

for all $t \in [0, T]$ and the complementary slackness condition

$$\lambda(t)[c - G(y(t), z(t); t)] = 0$$

for all $t \in [0, T]$, (13) and (14) form a system of 4 equations in 4 unknowns: $z(t)$, $y(t)$, $\pi(t)$, $\lambda(t)$.

An Alternative Formulation

The pair of differential equations (14)-(15) must be solved subject to 2 boundary conditions: the initial condition

$$y(0) \text{ given} \tag{16}$$

and the TVC

$$\phi[y(T) - y^*] = \pi(T)[y(T) - y^*] = 0 \tag{17}$$

or

$$\lim_{T \to \infty} \pi(T)[y(T) - y^*] = 0 \tag{18}$$

An Alternative Formulation

Alternatively, define the present value Hamiltonian

$$\begin{aligned} &\hat{H}(z(t), y(t), \pi(t); t) \\ &= e^{-\rho t} F(y(t), z(t); t) + \pi(t) Q(y(t), z(t); t) \end{aligned}$$

And the "maximized present value Hamiltonian"

$$\begin{aligned} H(y(t), \pi(t); t) = &\max_{z(t)} \hat{H}(z(t), y(t), \pi(t); t) \\ &\text{subject to } c \geq G(y(t), z(t); t) \end{aligned}$$

An Alternative Formulation

The maximized present value Hamiltonian

$$\begin{aligned} H(y(t), \pi(t); t) = \max_{z(t)} \; & e^{-\rho t} F(y(t), z(t); t) \\ & + \pi(t) Q(y(t), z(t); t) \\ & \text{subject to } c \geq G(y(t), z(t); t) \end{aligned} \tag{19}$$

is the maximum value function for a static, constrained optimization problem with a single choice variable and a single constraint.

An Alternative Formulation

By the envelope theorem

$$
\begin{aligned}
H(y(t), \pi(t); t) = \max_{z(t)} & \, e^{-\rho t} F(y(t), z(t); t) \\
& + \pi(t) Q(y(t), z(t); t) \\
& \text{subject to } c \geq G(y(t), z(t); t)
\end{aligned}
\tag{19}
$$

satisfies

$$
\begin{aligned}
H_y(y(t), \pi(t); t) = \; & e^{-\rho t} F_y(y(t), z(t); t) + \pi(t) Q_y(y(t), z(t); t) \\
& - \lambda(t) G(y(t), z(t); t)
\end{aligned}
\tag{20}
$$

and

$$
H_\pi(y(t), \pi(t); t) = Q(y(t), z(t); t) \tag{21}
$$

An Alternative Formulation

$$
\begin{aligned}
H(y(t), \pi(t); t) = \max_{z(t)} & \, e^{-\rho t} F(y(t), z(t); t) \\
& + \pi(t) Q(y(t), z(t); t) \\
& \text{subject to } c \geq G(y(t), z(t); t)
\end{aligned} \tag{19}
$$

The envelope theorem holds because $z(t)$ and $\lambda(t)$ satisfy the FOC

$$
\begin{aligned}
e^{-\rho t} F_z(y(t), z(t); t) + \pi(t) Q_z(y(t), z(t); t) & \\
-\lambda(t) G_z(y(t), z(t); t) &= 0
\end{aligned} \tag{22}
$$

and the complementary slackness condition

$$
\lambda(t)[c - G(y(t), z(t); t)] = 0
$$

for the static problem in (19).

An Alternative Formulation

But the FOC and complementary slackness condition

$$
\begin{aligned}
e^{-\rho t} F_z(y(t), z(t); t) + \pi(t) Q_z(y(t), z(t); t) \\
-\lambda(t) G_z(y(t), z(t); t) = 0
\end{aligned}
\qquad (22)
$$

$$
\lambda(t)[c - G(y(t), z(t); t)] = 0
$$

for the static problem coincide with the FOC and complementary slackness condition

$$
\begin{aligned}
e^{-\rho t} F_z(y(t), z(t); t) + \pi(t) Q_z(y(t), z(t); t) \\
-\lambda(t) G_z(y(t), z(t); t) = 0
\end{aligned}
\qquad (13)
$$

$$
\lambda(t)[c - G(y(t), z(t); t)] = 0
$$

for the dynamic problem.

An Alternative Formulation

And in light of

$$H_y(y(t), \pi(t); t) = e^{-\rho t} F_y(y(t), z(t); t) + \pi(t) Q_y(y(t), z(t); t) - \lambda(t) G(y(t), z(t); t) \tag{20}$$

and

$$H_\pi(y(t), \pi(t); t) = Q(y(t), z(t); t) \tag{21}$$

(14) and (15) can be written more compactly as (the Hamiltonian system)

$$\dot{\pi}(t) = -H_y(y(t), \pi(t); t) \tag{23}$$

$$\dot{y}(t) = H_\pi(y(t), \pi(t); t) \tag{24}$$

An Alternative Formulation

Theorem (Maximum Principle) Consider the continuous time dynamic optimization problem: Given $y(0)$, choose functions $z(t), t \in [0, T]$, and $y(t), t \in (0, T]$, to maximize

$$\int_0^T e^{-\rho t} F(y(t), z(t); t) dt$$

subject to the constraints

$$Q(y(t), z(t); t) \geq \dot{y}(t) \text{ for all } t \in [0, T]$$

$$c \geq G(y(t), z(t); t) \text{ for all } t \in [0, T]$$

$$y(T) \geq y^*$$

An Alternative Formulation

Associated with this problem, define the present value Hamiltonian

$$
\begin{aligned}
&\hat{H}(z(t), y(t), \pi(t); t) \\
&= e^{-\rho t} F(y(t), z(t); t) + \pi(t) Q(y(t), z(t); t)
\end{aligned}
$$

and the maximized present value Hamiltonian

$$
\begin{aligned}
H(y(t), \pi(t); t) = &\max_{z(t)} \hat{H}(z(t), y(t), \pi(t); t) \\
&\text{subject to } c \geq G(y(t), z(t); t)
\end{aligned}
\tag{19}
$$

Then the solution to the dynamic problem must satisfy ...

An Alternative Formulation

a) The FOC and complementary slackness condition for the static problem in (19):

$$e^{-\rho t}F_z(y(t), z(t); t) + \pi(t)Q_z(y(t), z(t); t) \\ -\lambda(t)G_z(y(t), z(t); t) = 0 \quad (22)$$

for all $t \in [0, T]$

$$\lambda(t)[c - G(y(t), z(t); t)] = 0$$

for all $t \in [0, T]$

An Alternative Formulation

b) The pair of differential equations:

$$\dot{\pi}(t) = -H_y(y(t), \pi(t); t) \tag{23}$$

for all $t \in [0, T]$

$$\dot{y}(t) = H_\pi(y(t), \pi(t); t) \tag{24}$$

for all $t \in [0, T]$

Where H_y and H_π can be computed using the envelope theorem applied to the static problem in (19).

An Alternative Formulation

c) The boundary conditions

$$y(0) \text{ given} \tag{16}$$

and

$$\pi(T)[y(T) - y^*] = 0 \tag{17}$$

or

$$\lim_{T\to\infty} \pi(T)[y(T) - y^*] = 0 \tag{18}$$