ECON 772001
MATH FOR ECONOMISTS

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October 27, 2020

Life Cycle Saving

A consumer works during periods $t = 0, 1, \ldots, T$, then retires.

w = constant labor income

Life Cycle Saving

k_t = bank account balance at the beginning of t

$k_0 = 0$ given

k_t can be negative for $t = 0, 1, \ldots, T$

Borrowing is allowed, but

$$k_{T+1} \geq k^* > 0$$

k^* = saving needed for retirement

Life Cycle Saving

$r =$ constant interest rate

$c_t =$ consumption

$$k_{t+1} = k_t + w + rk_t - c_t$$

Allowing for free disposal:

$$w + rk_t - c_t \geq k_{t+1} - k_t$$

for all $t = 0, 1, \ldots, T$

$$Q(k_t, c_t) = w + rk_t - c_t$$

Life Cycle Saving

The consumer's problem: Given $k_0 = 0$, choose sequences $\{c_t\}_{t=0}^{T}$ and $\{k_t\}_{t=1}^{T+1}$ to maximize

$$\sum_{t=0}^{T} \beta^t \ln(c_t)$$

subject to

$$w + rk_t - c_t \geq k_{t+1} - k_t$$

for all $t = 0, 1, \ldots, T$

$$k_{T+1} \geq k^*$$

Life Cycle Saving

Use the present value Hamiltontian

$$H(k_t, \pi_{t+1}; t) = \max_{c_t} \beta^t \ln(c_t) + \pi_{t+1}(w + rk_t - c_t)$$

to derive the optimality conditions

$$\frac{\beta^t}{c_t} - \pi_{t+1} = 0 \text{ for all } t = 0, 1, \ldots, T$$

$$\pi_{t+1} - \pi_t = -H_k(k_t, \pi_{t+1}; t) = -\pi_{t+1} r \text{ for all } t = 1, 2, \ldots, T$$

$$k_{t+1} - k_t = H_\pi(k_t, \pi_{t+1}; t) = w + rk_t - c_t \text{ for all } t = 0, 1, \ldots, T$$

Life Cycle Saving

Boundary conditions:

$$k_0 = 0 \text{ given}$$

$$\pi_{T+1}(k_{T+1} - k^*) = 0$$

Since the FOC for c_T implies

$$\pi_{T+1} = \frac{\beta^T}{c_T} > 0$$

the transversality condition requires

$$k_{T+1} = k^*$$

Here, as in the Ramsey model, the TVC prevents overaccumulation.

Life Cycle Saving

Three equations in three unknowns: c_t, k_t, and π_t

$$\frac{\beta^t}{c_t} - \pi_{t+1} = 0 \text{ for all } t = 0, 1, \dots, T$$

$$\pi_{t+1} - \pi_t = -\pi_{t+1} r \text{ for all } t = 1, 2, \dots, T$$

$$k_{t+1} - k_t = w + rk_t - c_t \text{ for all } t = 0, 1, \dots, T$$

Again as in the Ramsey model, let's use the FOC for c_t to eliminate π_t.

Life Cycle Saving

$$\pi_{t+1} - \pi_t = -\pi_{t+1} r$$

$$(1+r)\pi_{t+1} = \pi_t$$

$$\frac{(1+r)\beta^t}{c_t} = \frac{\beta^{t-1}}{c_{t-1}}$$

$$\frac{c_t}{\beta c_{t-1}} = 1 + r$$

The intertemporal marginal rate of substitution equals the slope of the intertemporal budget constraint.

Life Cycle Saving

$$c_t = \beta(1+r)c_{t-1}$$

$$c_1 = \beta(1+r)c_0$$

$$c_2 = \beta(1+r)c_1 = [\beta(1+r)]^2 c_0$$

$$c_t = [\beta(1+r)]^t c_0$$

for all $t = 0, 1, \ldots, T$

Life Cycle Saving

$$k_{t+1} - k_t = w + rk_t - c_t$$

$$k_{t+1} = (1+r)k_t + w - c_t$$

$$k_1 = (1+r)k_0 + w - c_0$$

$$\begin{aligned} k_2 &= (1+r)k_1 + w - c_1 \\ &= (1+r)^2 k_0 + (1+r)(w - c_0) + w - c_1 \end{aligned}$$

Life Cycle Saving

$$k_{t+1} = (1+r)k_t + w - c_t$$

$$k_1 = (1+r)k_0 + w - c_0$$

$$k_2 = (1+r)^2 k_0 + (1+r)(w - c_0) + w - c_1$$

$$\begin{aligned} k_3 &= (1+r)k_2 + w - c_2 \\ &= (1+r)^3 k_0 + (1+r)^2 (w - c_0) \\ &\quad + (1+r)(w - c_1) + w - c_2 \end{aligned}$$

Life Cycle Saving

$$k_1 = (1 + r)k_0 + w - c_0$$

$$k_2 = (1 + r)^2 k_0 + (1 + r)(w - c_0) + w - c_1$$

$$k_3 = (1 + r)^3 k_0 + (1 + r)^2 (w - c_0) + (1 + r)(w - c_1) + w - c_2$$

$$k_{T+1} = (1 + r)^{T+1} k_0 + \sum_{t=0}^{T} (1 + r)^{T-t} (w - c_t)$$

Life Cycle Saving

$$k_{T+1} = (1+r)^{T+1} k_0 + \sum_{t=0}^{T} (1+r)^{T-t} (w - c_t)$$

$$\frac{k_{T+1}}{(1+r)^T} = (1+r) k_0 + \sum_{t=0}^{T} \frac{w}{(1+r)^t} - \sum_{t=0}^{T} \frac{c_t}{(1+r)^t}$$

$$\frac{k_{T+1}}{(1+r)^T} = (1+r) k_0 + \sum_{t=0}^{T} \frac{w}{(1+r)^t} - \sum_{t=0}^{T} \frac{\beta^t (1+r)^t c_0}{(1+r)^t}$$

Life Cycle Saving

$$\frac{k_{T+1}}{(1+r)^T} = (1+r)k_0 + \sum_{t=0}^{T} \frac{w}{(1+r)^t} - \sum_{t=0}^{T} \frac{\beta^t (1+r)^t c_0}{(1+r)^t}$$

$$\frac{k_{T+1}}{(1+r)^T} = (1+r)k_0 + \sum_{t=0}^{T} \frac{w}{(1+r)^t} - c_0 \sum_{t=0}^{T} \beta^t$$

Now use the initial condition $k_0 = 0$ and TVC $k_{T+1} = k^*$:

$$\frac{k^*}{(1+r)^T} = \sum_{t=0}^{T} \frac{w}{(1+r)^t} - c_0 \sum_{t=0}^{T} \beta^t$$

Life Cycle Saving

$$\frac{k^*}{(1+r)^T} = \sum_{t=0}^{T} \frac{w}{(1+r)^t} - c_0 \sum_{t=0}^{T} \beta^t$$

Similar to what happens in the Ramsey model:

Choose c_0 too high and there's not enough saving for retirement.

Choose c_0 too low and there's too much saving for retirement.

Life Cycle Saving

$$\frac{k^*}{(1+r)^T} = \sum_{t=0}^{T} \frac{w}{(1+r)^t} - c_0 \sum_{t=0}^{T} \beta^t$$

Different from the Ramsey model, it is possible to solve explicitly for the optimal c_0:

$$c_0 = \left(\sum_{t=0}^{T} \beta^t \right)^{-1} \left[\sum_{t=0}^{T} \frac{w}{(1+r)^t} - \frac{k^*}{(1+r)^T} \right]$$

Life Cycle Saving

$$c_0 = \left(\sum_{t=0}^{T} \beta^t\right)^{-1} \left[\sum_{t=0}^{T} \frac{w}{(1+r)^t} - \frac{k^*}{(1+r)^T}\right]$$

$$\left(\sum_{t=0}^{T} \beta^t\right)^{-1} = \text{ a fraction}$$

$$\sum_{t=0}^{T} \frac{w}{(1+r)^t} = \text{ PV of labor income}$$

$$\frac{k^*}{(1+r)^T} = \text{ PV of retirement saving}$$

Life Cycle Saving

$$c_0 = \left(\sum_{t=0}^{T} \beta^t \right)^{-1} \left[\sum_{t=0}^{T} \frac{w}{(1+r)^t} - \frac{k^*}{(1+r)^T} \right]$$

Set c_0 as a fraction of "discretionary wealth," that is, the PV of labor income after saving for retirement.

$$c_t = [\beta(1+r)]^t c_0 \text{ for all } t = 0, 1, \ldots, T$$

Then consumption grows or shrinks depending on a comparison between patience β and the interest rate $1 + r$.

The Maximum Principle in Continuous Time

A Continuous Time Dynamic Optimization Problem

The Kuhn-Tucker Formulation

An Alternative Formulation

The Ramsey Model: Present and Current Value Hamiltonians

A Continuous Time Problem

Continuous time $t \in [0, T]$

$y(t) =$ stock variable at t

$z(t) =$ flow variable at t

A Continuous Time Problem

Objective function in discrete time

$$\sum_{t=0}^{T} \beta^t F(y_t, z_t; t)$$

Objective function in continuous time

$$\int_0^T e^{-\rho t} F(y(t), z(t); t) dt$$

$\rho \geq 0$ discount rate

$$e^{-\rho} = \beta \Rightarrow \rho = 0.05, \beta \approx 0.95$$

A Continuous Time Problem

Constraints governing the stock in discrete time ($\Delta t = 1$)

$$Q(y_t, z_t; t) \geq y_{t+1} - y_t$$

$$Q(y_t, z_t; t)\Delta t \geq y_{t+1} - y_t$$

In continuous time, $\Delta t \rightarrow 0$

$$Q(y(t), z(t); t) \geq \dot{y}(t)$$

for all $t \in [0, T]$

A Continuous Time Problem

Constraint on z_t given y_t in discrete time

$$c \geq G(y_t, z_t; t)$$

In continuous time

$$c \geq G(y(t), z(t); t)$$

for all $t \in [0, T]$

A Continuous Time Problem

Initial stock:

$$y(0) \text{ given}$$

Constraint on terminal stock:

$$y(T) \geq y^*$$

A Continuous Time Problem

The problem: Given $y(0)$, choose functions $z(t), t \in [0, T]$, and $y(t), t \in (0, T]$, to maximize

$$\int_0^T e^{-\rho t} F(y(t), z(t); t) dt$$

subject to the constraints

$$Q(y(t), z(t); t) \geq \dot{y}(t) \text{ for all } t \in [0, T]$$

$$c \geq G(y(t), z(t); t) \text{ for all } t \in [0, T]$$

$$y(T) \geq y^*$$

The Kuhn-Tucker Formulation

$$\begin{aligned} L(z(t), y(t), \pi(t), \lambda(t), \phi) = & \int_0^T e^{-\rho t} F(y(t), z(t); t) dt \\ & + \int_0^T \pi(t)[Q(y(t), z(t); t) - \dot{y}(t)] dt \\ & + \int_0^T \lambda(t)[c - G(y(t), z(t); t)] dt \\ & + \phi[y(T) - y^*] \end{aligned}$$

Use integration by parts to substitute out for the term involving $\pi(t)\dot{y}(t)$ in the "present value" Lagrangian.

The Kuhn-Tucker Formulation

$$\frac{d\pi(t)y(t)}{dt} = \dot{\pi}(t)y(t) + \pi(t)\dot{y}(t)$$

$$\int_0^T \left[\frac{d\pi(t)y(t)}{dt}\right] dt = \int_0^T \dot{\pi}(t)y(t)dt + \int_0^T \pi(t)\dot{y}(t)dt$$

$$\pi(T)y(T) - \pi(0)y(0) = \int_0^T \dot{\pi}(t)y(t)dt + \int_0^T \pi(t)\dot{y}(t)dt$$

$$-\int_0^T \pi(t)\dot{y}(t)dt = \int_0^T \dot{\pi}(t)y(t)dt + \pi(0)y(0) - \pi(T)y(T)$$

The Kuhn-Tucker Formulation

$$
\begin{aligned}
L(z(t), y(t), \pi(t), \lambda(t), \phi) = & \int_0^T e^{-\rho t} F(y(t), z(t); t) dt \\
& + \int_0^T \pi(t) Q(y(t), z(t); t) dt \\
& + \int_0^T \dot{\pi}(t) y(t) dt \\
& + \pi(0) y(0) - \pi(T) y(T) \\
& + \int_0^T \lambda(t) [c - G(y(t), z(t); t)] dt \\
& + \phi [y(T) - y^*]
\end{aligned}
$$