ECON 772001
MATH FOR ECONOMISTS

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A Discrete Time Problem

The problem: Given y_0, choose sequences $\{z_t\}_{t=0}^{T}$ and $\{y_t\}_{t=1}^{T+1}$ to maximize

$$\sum_{t=0}^{T} \beta^t F(y_t, z_t; t)$$

subject to the constraints

$$y_t + Q(y_t, z_t; t) \geq y_{t+1} \text{ for all } t = 0, 1, \ldots, T$$

$$c \geq G(y_t, z_t; t) \text{ for all } t = 0, 1, \ldots, T$$

$$y_{T+1} \geq y^*$$

The Kuhn-Tucker Formulation

$$
\begin{aligned}
&L(\{z_t\}_{t=0}^{T}, \{y_t\}_{t=1}^{T+1}, \{\pi_t\}_{t=1}^{T+1}, \{\lambda_t\}_{t=0}^{T}, \phi) \\
=&\sum_{t=0}^{T} \beta^t F(y_t, z_t; t) \\
&+ \sum_{t=0}^{T} \pi_{t+1}[y_t + Q(y_t, z_t; t) - y_{t+1}] \\
&+ \sum_{t=0}^{T} \lambda_t[c - G(y_t, z_t; t)] \\
&+ \phi(y_{T+1} - y^*)
\end{aligned}
$$

The Kuhn-Tucker Formulation

$$L = \sum_{t=0}^{T} \beta^t F(y_t, z_t; t) + \sum_{t=0}^{T} \pi_{t+1}[y_t + Q(y_t, z_t; t) - y_{t+1}]$$
$$+ \sum_{t=0}^{T} \lambda_t [c - G(y_t, z_t; t)] + \phi(y_{T+1} - y^*)$$

Note that L is in "present value" form.

To convert to "current values", use the change of variables $\pi_{t+1} = \beta^t \theta_{t+1}$ and $\lambda_t = \beta^T \mu_t$.

The Kuhn-Tucker Formulation

FOC for z_t:

$$\beta^t F_z(y_t, z_t; t) + \pi_{t+1} Q_z(y_t, z_t; t) - \lambda_t G_z(y_t, z_t; t) = 0 \quad (1)$$

for all $t = 0, 1, \ldots, T$

FOC for y_t:

$$\begin{aligned} &\pi_{t+1} - \pi_t \\ = &- [\beta^t F_y(y_t, z_t; t) + \pi_{t+1} Q_y(y_t, z_t; t) - \lambda_t G_y(y_t, z_t; t)] \end{aligned} \quad (2)$$

for all $t = 1, 2, \ldots, T$

The Kuhn-Tucker Formulation

Together with the binding constraints

$$y_{t+1} = y_t + Q(y_t, z_t; t) \tag{3}$$

for all $t = 0, 1, \ldots, T$ and the complementary slackness conditions

$$\lambda_t[c - G(y_t, z_t; t)] = 0$$

for all $t = 0, 1, \ldots, T$, (1) and (2) form a system of 4 equations in 4 unknowns: y_t, z_t, π_{t+1}, and λ_t.

The Kuhn-Tucker Formulation

Equations (2) and (3) form a pair of difference equations, which must be satisfied subject to two boundary conditions: the initial condition

$$y_0 \text{ given} \tag{4}$$

and the terminal, or transversality condition

$$\phi(y_{T+1} - y^*) = \pi_{T+1}(y_{T+1} - y^*) = 0 \tag{5}$$

or

$$\lim_{T \to \infty} \pi_{T+1}(y_{T+1} - y^*) = 0 \tag{6}$$

An Alternative Formulation

Alternatively, we can exploit the recursive structure of the problem.

At each $t = 0, 1, \ldots, T$, given y_t, choose z_t, taking into account its effects on $\beta^t F(y_t, z_t; t)$ and on y_{t+1} through $Q(y_t, z_t; t)$.

Then leave future decisions for future periods.

An Alternative Formulation

With these ideas in mind, define the present value Hamiltonian

$$\hat{H}(z_t, y_t, \pi_{t+1}; t) = \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t)$$

And the "maximized present value Hamiltonian"

$$H(y_t, \pi_{t+1}; t) = \max_{z_t} \hat{H}(z_t, y_t, \pi_{t+1}; t) \text{ subject to } c \geq G(y_t, z_t; t)$$

or

$$\begin{aligned} H(y_t, \pi_{t+1}; t) = \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t) \\ \text{subject to } c \geq G(y_t, z_t; t) \end{aligned} \tag{7}$$

An Alternative Formulation

The maximized present value Hamiltonian

$$
\begin{aligned}
H(y_t, \pi_{t+1}; t) = & \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t) \\
& \text{subject to } c \geq G(y_t, z_t; t)
\end{aligned} \tag{7}
$$

is the maximum value function for a static, constrained optimization problem with a single choice variable and a single constraint.

An Alternative Formulation

By the envelope theorem

$$H(y_t, \pi_{t+1}; t) = \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t) \\ \text{subject to } c \geq G(y_t, z_t; t) \tag{7}$$

satisfies

$$H_y(y_t, \pi_{t+1}; t) = \beta^t F_y(y_t, z_t; t) + \pi_{t+1} Q_y(y_t, z_t; t) \\ - \lambda_t G_y(y_t, z_t; t) \tag{8}$$

and

$$H_\pi(y_t, \pi_{t+1}; t) = Q(y_t, z_t; t) \tag{9}$$

An Alternative Formulation

$$\begin{aligned} H(y_t, \pi_{t+1}; t) = & \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t) \\ & \text{subject to } c \geq G(y_t, z_t; t) \end{aligned} \tag{7}$$

The envelope theorem holds because z_t and λ_t satisfy the FOC

$$\beta^t F_z(y_t, z_t; t) + \pi_{t+1} Q_z(y_t, z_t; t) - \lambda_t G_z(y_t, z_t; t) = 0 \tag{10}$$

and the complementary slackness condition

$$\lambda_t [c - G(y_t, z_t; t)] = 0$$

for the static problem in (7).

An Alternative Formulation

But the FOC and complementary slackness condition

$$\beta^t F_z(y_t, z_t; t) + \pi_{t+1} Q_z(y_t, z_t; t) - \lambda_t G_z(y_t, z_t; t) = 0 \quad (10)$$

$$\lambda_t [c - G(y_t, z_t; t)] = 0$$

for the static problem coincide with the FOC and complementary slackness condition

$$\beta^t F_z(y_t, z_t; t) + \pi_{t+1} Q_z(y_t, z_t; t) - \lambda_t G_z(y_t, z_t; t) = 0 \quad (1)$$

$$\lambda_t [c - G(y_t, z_t; t)] = 0$$

for the dynamic problem.

An Alternative Formulation

And in light of

$$H_y(y_t, \pi_{t+1}; t) = \beta^t F_y(y_t, z_t; t) + \pi_{t+1} Q_y(y_t, z_t; t) - \lambda_t G_y(y_t, z_t; t) \tag{8}$$

$$H_\pi(y_t, \pi_{t+1}; t) = Q(y_t, z_t; t) \tag{9}$$

(2) and (3) can be written more compactly as

$$\pi_{t+1} - \pi_t = -H_y(y_t, \pi_{t+1}; t) \tag{11}$$

$$y_{t+1} - y_t = H_\pi(y_t, \pi_{t+1}; t) \tag{12}$$

An Alternative Formulation

Theorem (Maximum Principle) Consider the discrete time dynamic optimization problem: Given y_0, choose sequences $\{z_t\}_{t=0}^{T}$ and $\{y_t\}_{t=1}^{T+1}$ to maximize

$$\sum_{t=0}^{T} \beta^t F(y_t, z_t; t)$$

subject to the constraints

$$y_t + Q(y_t, z_t; t) \geq y_{t+1} \text{ for all } t = 0, 1, \ldots, T$$

$$c \geq G(y_t, z_t; t) \text{ for all } t = 0, 1, \ldots, T$$

$$y_{T+1} \geq y^*$$

An Alternative Formulation

Associated with this problem, define the present value Hamiltonian

$$\hat{H}(z_t, y_t, \pi_{t+1}; t) = \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t)$$

and the maximized present value Hamiltonian

$$\begin{aligned} H(y_t, \pi_{t+1}; t) = \; & \max_{z_t} \hat{H}(z_t, y_t, \pi_{t+1}; t) \\ & \text{subject to } c \geq G(y_t, z_t; t) \end{aligned} \tag{7}$$

Then the solution to the dynamic problem must satisfy ...

An Alternative Formulation

a) The FOC and complementary slackness condition for the static problem in (7):

$$\beta^t F_z(y_t, z_t; t) + \pi_{t+1} Q_z(y_t, z_t; t) - \lambda_t G_z(y_t, z_t; t) = 0 \quad (10)$$

for all $t = 0, 1, \ldots, T$

$$\lambda_t [c - G(y_t, z_t; t)] = 0$$

for all $t = 0, 1, \ldots, T$

An Alternative Formulation

b) The pair of difference equations:

$$\pi_{t+1} - \pi_t = -H_y(y_t, \pi_{t+1}; t) \tag{11}$$

for all $t = 1, 2, \ldots, T$

$$y_{t+1} - y_t = H_\pi(y_t, \pi_{t+1}; t) \tag{12}$$

for all $t = 0, 1, \ldots, T$

Where H_y and H_π can be computed using the envelope theorem applied to the static problem in (7).

An Alternative Formulation

c) The boundary conditions

$$y_0 \text{ given} \tag{4}$$

and

$$\pi_{T+1}(y_{T+1} - y^*) = 0 \tag{5}$$

or

$$\lim_{T \to \infty} \pi_{T+1}(y_{T+1} - y^*) = 0 \tag{6}$$

An Alternative Formulation

See the notes on "Euler Equations and Transversality Conditions" for:

A proof of the necessity of the transversality condition for a natural resource depletion problem.

A sketch of the proof of the sufficiency of the transversality condition for a more general discrete-time dynamic optimization problem with a concave objective.

Optimal Growth

Consider a discrete-time version of the Ramsey model with log utility, Cobb-Douglas production, and an infinite horizon.

The social planner's problem: Given k_0, choose sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$k_t^{\alpha} - \delta k_t - c_t \geq k_{t+1} - k_t$$

for all $t = 0, 1, 2, \ldots$.

Optimal Growth

Given k_0, choose sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$k_t^{\alpha} - \delta k_t - c_t \geq k_{t+1} - k_t$$

for all $t = 0, 1, 2, \ldots$.

For this problem, $y_t = k_t$, $z_t = c_t$,

$$F(k_t, c_t; t) = F(c_t) = \ln(c_t)$$

$$Q(k_t, c_t; t) = Q(k_t, c_t) = k_t^{\alpha} - \delta k_t - c_t$$

Optimal Growth

Given k_0, choose sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$k_t^{\alpha} - \delta k_t - c_t \geq k_{t+1} - k_t$$

for all $t = 0, 1, 2, \ldots$.

The maximized present value Hamiltonian is

$$H(k_t, \pi_{t+1}; t) = \max_{c_t} \ \beta^t \ln(c_t) + \pi_{t+1}(k_t^{\alpha} - \delta k_t - c_t)$$

Optimal Growth

$$H(k_t, \pi_{t+1}; t) = \max_{c_t} \ \beta^t \ln(c_t) + \pi_{t+1}(k_t^\alpha - \delta k_t - c_t)$$

FOC for c_t:

$$\frac{\beta^t}{c_t} - \pi_{t+1} = 0$$

for all $t = 0, 1, 2, \ldots$

Optimal Growth

$$H(k_t, \pi_{t+1}; t) = \max_{c_t} \; \beta^t \ln(c_t) + \pi_{t+1}(k_t^\alpha - \delta k_t - c_t)$$

Difference equations for k_t and π_t:

$$\pi_{t+1} - \pi_t = -H_k(k_t, \pi_{t+1}; t) = -\pi_{t+1}(\alpha k_t^{\alpha-1} - \delta)$$

for all $t = 1, 2, 3, \ldots$

$$k_{t+1} - k_t = H_\pi(k_t, \pi_{t+1}; t) = k_t^\alpha - \delta k_t - c_t$$

for all $t = 0, 1, 2, \ldots$

Optimal Growth

Boundary conditions:

$$k_0 \text{ given}$$

$$\lim_{T\to\infty} \pi_{T+1} k_{T+1} = 0$$

If, instead, we used the current value Hamiltonian, the TVC would be

$$\lim_{T\to\infty} \beta^T \theta_{T+1} k_{T+1} = 0$$

where $\pi_{T+1} = \beta^T \theta_{T+1}$.

Optimal Growth

Three equations in three unknowns: c_t, k_t, and π_t.

$$\frac{\beta^t}{c_t} - \pi_{t+1} = 0$$

$$\pi_{t+1} - \pi_t = -\pi_{t+1}(\alpha k_t^{\alpha-1} - \delta)$$

$$k_{t+1} - k_t = k_t^\alpha - \delta k_t - c_t$$

Let's use the FOC for c_t to eliminate π_{t+1}.

Optimal Growth

$$\frac{\beta^t}{c_t} - \pi_{t+1} = 0$$

$$\pi_{t+1} - \pi_t = -\pi_{t+1}(\alpha k_t^{\alpha-1} - \delta)$$

$$\pi_{t+1}(\alpha k_t^{\alpha-1} + 1 - \delta) = \pi_t$$

$$\frac{\beta^t}{c_t}(\alpha k_t^{\alpha-1} + 1 - \delta) = \frac{\beta^{t-1}}{c_{t-1}}$$

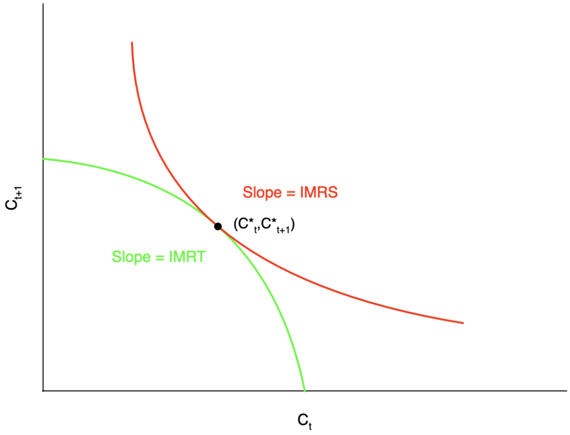
Optimal Growth

$$\frac{\beta^t}{c_t}(\alpha k_t^{\alpha-1} + 1 - \delta) = \frac{\beta^{t-1}}{c_{t-1}}$$

$$\frac{c_t}{\beta c_{t-1}} = \alpha k_t^{\alpha-1} + 1 - \delta$$

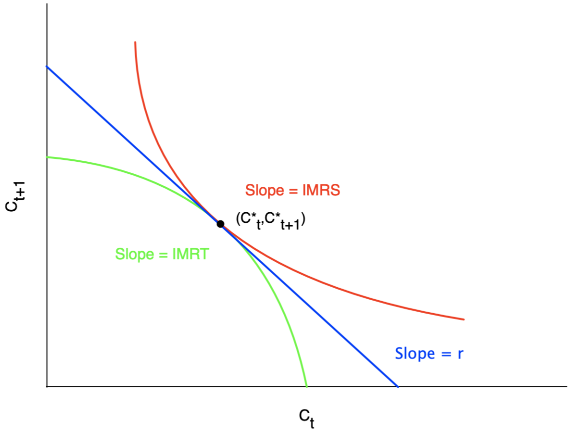
The left-hand side measures the intertemporal marginal rate of substitution, and the right-hand side measures the intertemporal marginal rate of transformation.

Optimal Growth



Pareto optimal allocations in the Ramsey model.

Optimal Growth



Competitive equilibrium allocations in the Ramsey model.

Optimal Growth

$$\frac{\beta^t}{c_t}(\alpha k_t^{\alpha-1} + 1 - \delta) = \frac{\beta^{t-1}}{c_{t-1}}$$

$$c_t = \beta(\alpha k_t^{\alpha-1} + 1 - \delta)c_{t-1}$$

for all $t = 1, 2, 3, \ldots$

$$c_{t+1} = \beta(\alpha k_{t+1}^{\alpha-1} + 1 - \delta)c_t$$

for all $t = 0, 1, 2, \ldots$

Optimal Growth

Two equations in two unknowns: c_t and k_t.

$$k_{t+1} - k_t = k_t^{\alpha} - \delta k_t - c_t \tag{36}$$

for all $t = 0, 1, 2, \ldots$

$$c_{t+1} = \beta(\alpha k_{t+1}^{\alpha-1} + 1 - \delta)c_t \tag{37}$$

or all $t = 0, 1, 2, \ldots$

Optimal Growth

$$k_{t+1} - k_t = k_t^{\alpha} - \delta k_t - c_t \tag{36}$$

$$c_{t+1} = \beta(\alpha k_{t+1}^{\alpha-1} + 1 - \delta)c_t \tag{37}$$

Given k_0, guess c_0.

Use (36) to compute k_1 and (37) to compute c_1.

Use (36) to compute k_2 and (37) to compute c_2.

Recursively construct candidate solutions $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$.

Optimal Growth

$$k_{t+1} - k_t = k_t^{\alpha} - \delta k_t - c_t \quad (36)$$

$$c_{t+1} = \beta(\alpha k_{t+1}^{\alpha-1} + 1 - \delta)c_t \quad (37)$$

Given k_0, guess c_0, then recursively construct candidate solutions $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$.

If your guess for c_0 is too high, k_t becomes negative.

If your guess for c_0 is too low, the TVC is violated.

Somewhere in between is the unique value of c_0^* that puts the economy on the stable manifold, converging to the steady state.

Optimal Growth

$$c_{t+1} = \beta(\alpha k_{t+1}^{\alpha-1} + 1 - \delta)c_t \tag{37}$$

In the steady state, $c_{t+1} = c_t = c^*$ and $k_{t+1} = k^*$, where

$$c^* = \beta[\alpha(k^*)^{\alpha-1} + 1 - \delta]c^*$$

$$k^* = \left[\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right]^{1/(\alpha-1)}$$

Optimal Growth

$$k_{t+1} - k_t = k_t^{\alpha} - \delta k_t - c_t \tag{36}$$

In the steady state, $c_t = c^*$ and $k_{t+1} = k_t = k^*$, where

$$0 = (k^*)^{\alpha} - \delta k^* - c^*$$

$$c^* = (k^*)^{\alpha} - \delta k^*$$