ECON 772001
MATH FOR ECONOMISTS

Peter Ireland

Boston College

October 20, 2020

These lecture notes by Peter Ireland are licensed under a Creative Commons Attribution-NonCommerical-ShareAlike 4.0 International (CC BY-NC-SA 4.0) License. http://creativecommons.org/licenses/by-nc-sa/4.0/.

The Maximum Principle in Discrete Time

Basic Elements of Dynamic Optimization Problems

The Maximum Principle on Discrete Time

- A Discrete Time Dynamic Optimization Problem
- The Kuhn-Tucker Formulation
- An Alternative Formulation

Examples:

- Optimal Growth
- Life Cycle Saving

The Maximum Principle in Discrete Time

References:

Dixit Ch 10

Acemoglu Ch 7

Pontryagin et al. *The Mathematical Theory of Optimal Processes*, 1962.

Basic Elements of Dynamic Problems

1. Variables indexed by t.

Large numbers of choice variables and constraints.

Basic Elements of Dynamic Problems

2. Distinction between stocks and flows.

Suggests using stocks as “state” variables.

Basic Elements of Dynamic Problems

3. Constraints linking stocks and flows.

Summarizing intertemporal trade-offs.

Even in discrete-time, these considerations may motivate us to prefer the Hamiltonian to the Lagrangian.

A Discrete Time Problem

Discrete time $t = 0, 1, \ldots, T$

y_t = stock variable at the beginning of t

z_t = flow variable during t

A Discrete Time Problem

Objective function:

$$\sum_{t=0}^{T} \beta^t F(y_t, z_t; t)$$

$\beta =$ discount factor, $0 < \beta \leq 1$

$$e^{-\rho} = \beta \Rightarrow \rho = 0.05, \beta \approx 0.95$$

A Discrete Time Problem

Constraints governing the stock:

$$Q(y_t, z_t; t) \geq y_{t+1} - y_t \text{ for all } t = 0, 1, \ldots, T$$

$$y_t + Q(y_t, z_t; t) \geq y_{t+1} \text{ for all } t = 0, 1, \ldots, T$$

Constraint on z_t given y_t:

$$c \geq G(y_t, z_t; t) \text{ for all } t = 0, 1, \ldots, T$$

A Discrete Time Problem

Initial stock:

$$y_0 \text{ given}$$

Constraint on terminal stock:

$$y_{T+1} \geq y^*$$

A Discrete Time Problem

The problem: Given y_0, choose sequences $\{z_t\}_{t=0}^{T}$ and $\{y_t\}_{t=1}^{T+1}$ to maximize

$$\sum_{t=0}^{T} \beta^t F(y_t, z_t; t)$$

subject to the constraints

$$y_t + Q(y_t, z_t; t) \geq y_{t+1} \text{ for all } t = 0, 1, \ldots, T$$

$$c \geq G(y_t, z_t; t) \text{ for all } t = 0, 1, \ldots, T$$

$$y_{T+1} \geq y^*$$

A Discrete Time Problem

The problem has many choice variables and constraints.

If $T < \infty$, however, this problem is a special case of not an extension to, the one for which we've already proved the Kuhn-Tucker theorem.

Additive time separability of the objective function and the recursive form of the constraints will be exploited by the maximum principle, but are not needed for an application of the Kuhn-Tucker theorem.

The Kuhn-Tucker Formulation

$$
\begin{aligned}
& L(\{z_t\}_{t=0}^{T}, \{y_t\}_{t=1}^{T+1}, \{\pi_t\}_{t=1}^{T+1}, \{\lambda_t\}_{t=0}^{T}, \phi) \\
= & \sum_{t=0}^{T} \beta^t F(y_t, z_t; t) \\
& + \sum_{t=0}^{T} \pi_{t+1}[y_t + Q(y_t, z_t; t) - y_{t+1}] \\
& + \sum_{t=0}^{T} \lambda_t[c - G(y_t, z_t; t)] \\
& + \phi(y_{T+1} - y^*)
\end{aligned}
$$

The Kuhn-Tucker Formulation

$$L = \sum_{t=0}^{T} \beta^t F(y_t, z_t; t) + \sum_{t=0}^{T} \pi_{t+1}[y_t + Q(y_t, z_t; t) - y_{t+1}]$$
$$+ \sum_{t=0}^{T} \lambda_t [c - G(y_t, z_t; t)] + \phi(y_{T+1} - y^*)$$

Note that L is in "present value" form.

To convert to "current values", use the change of variables $\pi_{t+1} = \beta^t \theta_{t+1}$ and $\lambda_t = \beta^t \mu_t$.

The Kuhn-Tucker Formulation

$$L = \sum_{t=0}^{T} \beta^t F(y_t, z_t; t) + \sum_{t=0}^{T} \pi_{t+1}[y_t + Q(y_t, z_t; t) - y_{t+1}]$$
$$+ \sum_{t=0}^{T} \lambda_t [c - G(y_t, z_t; t)] + \phi(y_{T+1} - y^*)$$

Fix $t = 0, 1, \ldots, T$. FOC for z_t:

$$\beta^t F_z(y_t, z_t; t) + \pi_{t+1} Q_z(y_t, z_t; t) - \lambda_t G_z(y_t, z_t; t) = 0 \quad (1)$$

for all $t = 0, 1, \ldots, T$

The Kuhn-Tucker Formulation

$$\sum_{t=0}^{T} \pi_{t+1}[y_t + Q(y_t, z_t; t) - y_{t+1}]$$

$$= \pi_1[y_0 + Q(y_0, z_0; 0) - y_1] + \pi_2[y_1 + Q(y_1, z_1; 1) - y_2]$$
$$+ \pi_3[y_2 + Q(y_2, z_2; 2) - y_3] + \pi_4[y_3 + Q(y_3, z_3; 3) - y_4] + \dots$$

For any $t = 1, 2, \dots, T$, the terms in the sum involving y_t are

$$\pi_{t+1}[y_t + Q(y_t, z_t; t)] - \pi_t y_t$$

The Kuhn-Tucker Formulation

$$L = \sum_{t=0}^{T} \beta^t F(y_t, z_t; t) + \sum_{t=0}^{T} \pi_{t+1}[y_t + Q(y_t, z_t; t) - y_{t+1}]$$
$$+ \sum_{t=0}^{T} \lambda_t [c - G(y_t, z_t; t)] + \phi(y_{T+1} - y^*)$$

Fix $t = 1, 2, \ldots, T$. FOC for y_t:

$$\beta^t F_y(y_t, z_t; t) + \pi_{t+1}[1 + Q_y(y_t, z_t; t)] - \pi_t - \lambda_t G_y(y_t, z_t; t) = 0$$

$$\begin{aligned} &\pi_{t+1} - \pi_t \\ &= -\left[\beta^t F_y(y_t, z_t; t) + \pi_{t+1} Q_y(y_t, z_t; t) - \lambda_t G_y(y_t, z_t; t)\right] \end{aligned} \tag{2}$$

for all $t = 1, 2, \ldots, T$

The Kuhn-Tucker Formulation

$$L = \sum_{t=0}^{T} \beta^t F(y_t, z_t; t) + \sum_{t=0}^{T} \pi_{t+1}[y_t + Q(y_t, z_t; t) - y_{t+1}]$$
$$+ \sum_{t=0}^{T} \lambda_t [c - G(y_t, z_t; t)] + \phi(y_{T+1} - y^*)$$

FOC for y_{T+1}:

$$-\pi_{T+1} + \phi = 0$$

$$\phi = \pi_{T+1}$$

The Kuhn-Tucker Formulation

Together with the binding constraints

$$y_{t+1} = y_t + Q(y_t, z_t; t) \tag{3}$$

for all $t = 0, 1, \ldots, T$ and the complementary slackness conditions

$$\lambda_t[c - G(y_t, z_t; t)] = 0$$

for all $t = 0, 1, \ldots, T$, (1) and (2) form a system of 4 equations in 4 unknowns: y_t, z_t, π_{t+1}, and λ_t.

The Kuhn-Tucker Formulation

Equations (2) and (3) form a pair of difference equations, which must be satisfied subject to two boundary conditions: the initial condition

$$y_0 \text{ given} \tag{4}$$

and the terminal, or transversality condition

$$\phi(y_{T+1} - y^*) = \pi_{T+1}(y_{T+1} - y^*) = 0 \tag{5}$$

or

$$\lim_{T \to \infty} \pi_{T+1}(y_{T+1} - y^*) = 0 \tag{6}$$