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MATH FOR ECONOMISTS

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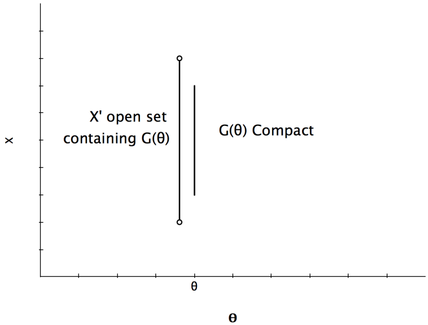
Continuity Concepts for Correspondences

The correspondence G is *upper hemicontinuous* at $\theta \in \Theta$ if $G(\theta)$ is nonempty and if, for every open set $X' \subseteq X$ with $G(\theta) \subseteq X'$ (every open subset X' of X containing $G(\theta)$), there exists a $\delta > 0$ such that for every $\theta' \in N_\delta(\theta)$ (every θ' in some δ-neighborhood of θ), $G(\theta') \subseteq X'$ ($G(\theta')$ is contained in X' as well).

Continuity Concepts for Correspondences



Take any θ' near θ. $G(\theta')$ must also be in X'. G can't suddenly "explode."

Continuity Concepts for Correspondences

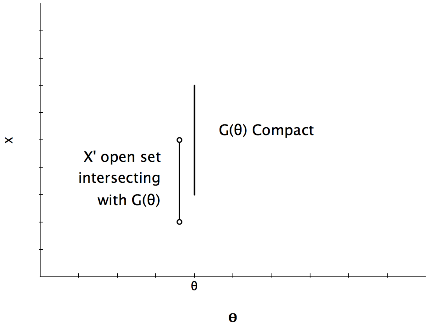
The *compact valued* correspondence G is *upper hemicontinuous* at $\theta \in \Theta$ if $G(\theta)$ is nonempty and if, for every sequence $\{\theta_j\}$ with $\theta_j \to \theta$ and every sequence $\{x_j\}$ with $x_j \in G(\theta_j)$ for all j, there exists a convergent subsequence $\{x_{j_k}\}$ such that $x_{j_k} \to x \in G(\theta)$.

If G is not compact valued, then these conditions are sufficient, but not necessary, for G to be upper hemicontinuous.

Continuity Concepts for Correspondences

The correspondence G is *lower hemicontinuous* at $\theta \in \Theta$ if $G(\theta)$ is nonempty and if, for every open set $X' \subseteq X$ with $G(\theta) \cap X' \neq \emptyset$ (every open subset X' of X intersecting with $G(\theta)$), there exists a $\delta > 0$ such that for every $\theta' \in N_\delta(\theta)$, $G(\theta') \cap X' \neq \emptyset$ (some δ-neighborhood of θ such that $G(\theta')$ also intersects with X' for every θ' in that neighborhood).

Continuity Concepts for Correspondences



Take any θ' near θ. $G(\theta')$ must also intersect with X'. G can't suddenly "implode."

Continuity Concepts for Correspondences

The correspondence G is *lower hemicontinuous* at $\theta \in \Theta$ if $G(\theta)$ is nonempty and if, for every $x \in G(\theta)$ and every sequence $\{\theta_j\}$ such that $\theta_j \to \theta$, there is a $J \geq 1$ and a sequence $\{x_j\}$ such that $x_j \in G(\theta_j)$ for all $j \geq J$ and $x_j \to x$.

The correspondence G is *continuous* at $\theta \in \Theta$ if it is both upper and lower hemicontinuous at θ. The correspondence G is *continuous* if it is continuous at every $\theta \in \Theta$.

Berge's Maximum Theorem

Let $X \subseteq \mathbb{R}^n$ and $\Theta \subseteq \mathbb{R}^m$. Let $F : X \times \Theta \to \mathbb{R}$ be a continuous function, and let $G : \Theta \to X$ be a compact valued and continuous correspondence. Then the maximum value function

$$V(\theta) = \max_{x \in G(\theta)} F(x, \theta)$$

is well-defined and continuous, and the optimal policy correspondence

$$x^*(\theta) = \{x \in G(\theta) \mid F(x, \theta) = V(\theta)\}$$

is nonempty, compact valued, and upper hemicontinuous.

Berge's Maximum Theorem

To prove the theorem, fix $\theta \in \Theta$.

Note first that since $G(\theta)$ is nonempty and compact, and since $F(\cdot, \theta)$ is continuous, Weierstrass' extreme value theorem implies that $V(\theta)$ is well-defined and that $x^*(\theta)$ is nonempty.

Next, we will show that $x^*(\theta)$ is compact valued.

Berge's Maximum Theorem

Since $x^*(\theta) \subseteq G(\theta)$ and $G(\theta)$ is compact, it follows that $x^*(\theta)$ is bounded.

Let $\{x_j\}$ be a sequence with $x_j \in x^*(\theta)$ for all j and $x_j \to x$. Since $G(\theta)$ is closed, it must be that $x \in G(\theta)$. And since $V(\theta) = F(x_j, \theta)$ for all j and F is continuous, it follows that $F(x, \theta) = V(\theta)$. Hence, $x \in x^*(\theta)$, so $x^*(\theta)$ is closed.

Therefore, by the Heine-Borel theorem, $x^*(\theta)$ is compact valued.

Berge's Maximum Theorem

Now we will show that $x^*(\theta)$ is upper hemicontinuous.

Fix $\theta \in \Theta$ and let $\{\theta_j\}$ be any sequence with $\theta_j \to \theta$. Then let $\{x_j\}$ be a sequence with $x_j \in x^*(\theta_j)$ for all j.

Is there a convergent subsequence with $x_{j_k} \to x \in x^*(\theta)$?

Berge's Maximum Theorem

Yes! Since G is upper hemicontinuous, there exists a subsequence $\{x_{j_k}\}$ converging to $x \in G(\theta)$.

Now, let $x' \in G(\theta)$ as well. Since G is lower hemicontinuous, there exists a sequence $\{x'_{j_k}\}$ with $x'_{j_k} \in G(\theta_{j_k})$ and $x'_{j_k} \to x'$. And since $F(x_{j_k}, \theta_{j_k}) \geq F(x'_{j_k}, \theta_{j_k})$ for all k and F is continuous, it follows that $F(x, \theta) \geq F(x', \theta)$. Moreover, this condition holds for *any* $x' \in G(\theta)$.

It follows that $x \in x^*(\theta)$, so that x^* is upper hemicontinuous.

Berge's Maximum Theorem

Finally, we will show that $V(\theta)$ is continuous.

Fix $\theta \in \Theta$, and let $\{\theta_j\}$ be any sequence with $\theta_j \to \theta$. Choose $x_j \in x^*(\theta_j)$ for all j and let

$$\bar{v} = \limsup_j V(\theta_j) = \lim_{j\to\infty} \left[\sup_{k \geq j} V(\theta_k) \right]$$

and

$$\underline{v} = \liminf_j V(\theta_j) = \lim_{j\to\infty} \left[\inf_{k \geq j} V(\theta_k) \right]$$

Berge's Maximum Theorem

Then there exists a subsequence $\{x_{j_k}\}$ such that

$$\bar{v} = \lim_k F(x_{j_k}, \theta_{j_k}).$$

But since x^* is upper hemicontinuous, these exists a subsequence of $\{x_{j_{k_l}}\}$ of $\{x_{j_k}\}$ converging to $x \in x^*(\theta)$

Hence

$$\bar{v} = \lim_l F(x_{j_{k_l}}, \theta_{j_{k_l}}) = F(x, \theta) = V(\theta).$$

Berge's Maximum Theorem

There also exists a subsequence $\{x_{j_k}\}$ such that

$$\underline{v} = \lim_k F(x_{j_k}, \theta_{j_k}).$$

But since x^* is upper hemicontinuous, these exists a subsequence of $\{x_{j_{k_l}}\}$ of $\{x_{j_k}\}$ converging to $x \in x^*(\theta)$

Hence

$$\underline{v} = \lim_l F(x_{j_{k_l}}, \theta_{j_{k_l}}) = F(x, \theta) = V(\theta).$$

Therefore, $V(\theta_j) \to V(\theta)$, completing the proof.

Example: Utility Maximization

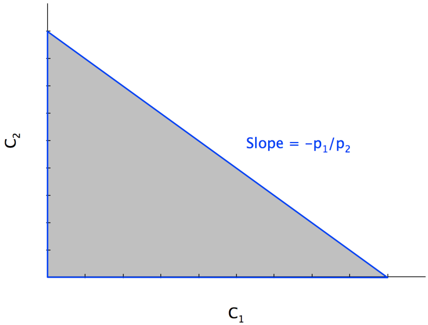
$$\max_{c_1, c_2} U(c_1, c_2) \text{ subject to } I \geq p_1 c_1 + p_2 c_2, c_1 \geq 0, c_2 \geq 0.$$

To apply Berge's theorem, we need U to be continuous and

$$G(I, p_1, p_2) = \{(c_1, c_2) \in \mathbb{R}^2 \mid I \geq p_1 c_1 + p_2 c_2, c_1 \geq 0, c_2 \geq 0\}$$

to be compact valued and continuous.

Example: Utility Maximization



Clearly, for any $I > 0$, $p_1 > 0$, and $p_2 > 0$, G is nonempty and compact valued. We only need to show that G satisfies the remaining requirements for upper and lower hemicontinuity.

Upper Hemicontinuity

Fix $(I, p_1, p_2) \in \mathbb{R}^3_{++}$, and let $\{I_j, p_{1j}, p_{2j}\}$ be a sequence in $\mathbb{R}^3_{++}$ with $(I_j, p_{1j}, p_{2j}) \to (I, p_1, p_2)$.

Since G is non-empty, there is a sequence $\{c_{1j}, c_{2j}\}$ with $(c_{1j}, c_{2j}) \in G(I_j, p_{1j}, p_{2j})$ for all j.

Is there a convergent subsequence $\{c_{1j_k}, c_{2j_k}\}$ with limit point $(c_1, c_2) \in G(I, p_1, p_2)$?

Upper Hemicontinuity

Yes! Since $(I_j, p_{1j}, p_{2j}) \rightarrow (I, p_1, p_2)$, there is a closed and bounded set $\hat{\Theta} \subseteq \mathbb{R}^3_{++} \subseteq \mathbb{R}^3$, such that, for some $J \geq 1$, all of the (I_j, p_{1j}, p_{2j}), $j \geq J$, and (I, p_1, p_2) are contained in $\hat{\Theta}$.

Moreover, the structure of G in this case implies that all of the $(c_{1j}, c_{2j}) \in G(I_j, p_{1j}, p_{2j})$ for $j \geq J$ will lie in a closed and bounded subset of $\mathbb{R}^2$.

Upper Hemicontinuity

Thus, for all $j \geq J$, all elements of the sequence $\{I_j, p_{1j}, p_{2j}, c_{1j}, c_{2j}\}$ lie in a closed and bounded subset of $\mathbb{R}^5$.

By the Bolzano-Weierstrass theorem, this sequence has a convergent subsequence $\{I_{j_k}, p_{1j_k}, p_{2j_k}, c_{1j_k}, c_{2k_k}\}$ with limit point (I, p_1, p_2, c_1, c_2). And since each element of this convergent subsequence satisfies

$$I_{j_k} \geq p_{1j_k} c_{1j_k} + p_{2j_k} c_{2j_k}, \; c_{1j_k} \geq 0, \text{ and } c_{2j_k} \geq 0,$$

it is easy to see that the limit point will also have to satisfy $(c_1, c_2) \in G(I, p_1, p_2)$.

Lower Hemicontinuity

Fix $(I, p_1, p_2) \in \mathbb{R}^3_{++}$ and $(c_1, c_2) \in G(I, p_1, p_2)$.

Then let $\{I_j, p_{1j}, p_{2j}\}$ be a sequence in $\mathbb{R}^3_{++}$ with $(I_j, p_{1j}, p_{2j}) \to (I, p_1, p_2)$.

Is there a sequence $\{c_{1j}, c_{2j}\}$ with $(c_{1j}, c_{2j}) \in G(I_j, p_{1j}, p_{2j})$ for all j and $(c_{1j}, c_{2j}) \to (c_1, c_2)$?

Yes! If $c_1 = c_2 = 0$, then $\{c_{1j}, c_{2j}\} = \{0, 0\}$ works.

Lower Hemicontinuity

So suppose that $c_1 > 0$ and/or $c_2 > 0$. Construct $\{c_{1j}, c_{2j}\}$ with

$$c_{1j} = \left(\frac{I_j}{I}\right)\left(\frac{p_1 c_1 + p_2 c_2}{p_{1j} c_1 + p_{2j} c_2}\right) c_1$$

and

$$c_{2j} = \left(\frac{I_j}{I}\right)\left(\frac{p_1 c_1 + p_2 c_2}{p_{1j} c_1 + p_{2j} c_2}\right) c_2.$$

Clearly, $c_{1j} \geq 0$ and $c_{2j} \geq 0$ for all j.

Lower Hemicontinuity

In addition

$$\begin{aligned} p_{1j}c_{1j} + p_{2j}c_{2j} &= \left(\frac{I_j}{I}\right)\left(\frac{p_1c_1 + p_2c_2}{p_{1j}c_1 + p_{2j}c_2}\right)(p_{1j}c_1 + p_{2j}c_2) \\ &= I_j\left(\frac{p_1c_1 + p_2c_2}{I}\right) \leq I_j, \end{aligned}$$

so that $(c_{1j}, c_{2j}) \in G(I_j, p_{1j}, p_{2j})$ for all j.

Moreover, $(c_{1j}, c_{2j}) \to (c_1, c_2)$, completing the proof.

Example: Utility Maximization

For the problem

$$\max_{c_1, c_2} U(c_1, c_2) \text{ subject to } I \geq p_1 c_1 + p_2 c_2, c_1 \geq 0, c_2 \geq 0,$$

we now know that if U is continuous:

1. $V(I, p_1, p_2)$ is well-defined and continuous.
2. $c_1^*(I, p_1, p_2)$ and $c_2^*(I, p_1, p_2)$ are nonempty, compact valued, and upper hemicontinuous correspondences.

And if we assume as well that U is strictly concave, the Marshallian (Walrasian) demand functions $c_1^*(I, p_1, p_2)$ and $c_2^*(I, p_1, p_2)$ are single-valued and continuous.