ECON 772001
MATH FOR ECONOMISTS

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Optimal Growth

Theorem (Maximum Principle) Consider the continuous time dynamic optimization problem: Given $k(0)$, choose $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ to maximize

$$\int_0^\infty e^{-\rho t} \ln(c(t)) dt$$

subject to

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t) \text{ for all } t \in [0, \infty)$$

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Associated with this problem, define the CV Hamiltonian

$$\tilde{H}(c(t), k(t), \theta(t)) = \ln(c(t)) + \theta(t)[k(t)^\alpha - \delta k(t) - c(t)]$$

and the maximized CV Hamiltonian

$$\begin{aligned} H(k(t), \theta(t)) = & \max_{c(t)} \ \tilde{H}(c(t), k(t), \theta(t)) \\ & \max_{c(t)} \ \ln(c(t)) + \theta(t)[k(t)^\alpha - \delta k(t) - c(t)] \end{aligned}$$

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Then the solution to the dynamic problem must satisfy: the FOC for the static problem

$$H(k(t), \theta(t)) = \max_{c(t)} \; \ln(c(t)) + \theta(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

$$\frac{1}{c(t)} - \theta(t) = 0$$

for all $t \in [0, \infty)$.

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The pair of differential equations

$$\dot{\theta}(t) = \rho\theta(t) - H_k(k(t), \theta(t))$$

$$\dot{k}(t) = H_\theta(k(t), \theta(t))$$

where the envelope theorem implies

$$\dot{\theta}(t) = \rho\theta(t) - \theta(t)[\alpha k(t)^{\alpha-1} - \delta]$$

$$\dot{k}(t) = k(t)^\alpha - \delta k(t) - c(t)$$

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The initial condition

$$k(0) \text{ given}$$

The transversality condition

$$\lim_{T\to\infty} e^{-\rho T}\theta(T)k(T) = 0$$

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Unlike linear differential equations, describing exponential growth or decay, the nonlinear system describing the solution to the Ramsey model lacks a closed-form solution.

But we can deduce the qualitative properties of the solution using a phase diagram, or solve the system numerically.

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The FOC and differential equations

$$\frac{1}{c(t)} - \theta(t) = 0$$

$$\dot{\theta}(t) = \rho\theta(t) - \theta(t)[\alpha k(t)^{\alpha-1} - \delta]$$
$$\dot{k}(t) = k(t)^{\alpha} - \delta k(t) - c(t)$$

form a three-dimensional system, which is difficult to depict graphically.

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Use the FOC to eliminate $\theta(t)$ from the system:

$$\frac{1}{c(t)} = \theta(t)$$

$$1 = \theta(t)c(t)$$

Differentiate with respect to t to obtain

$$0 = \dot{\theta}(t)c(t) + \theta(t)\dot{c}(t)$$

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Now substitute the differential equation

$$\dot{\theta}(t) = \rho\theta(t) - \theta(t)[\alpha k(t)^{\alpha-1} - \delta]$$

into

$$0 = \dot{\theta}(t)c(t) + \theta(t)\dot{c}(t)$$

to obtain

$$0 = \{\rho\theta(t) - \theta(t)[\alpha k(t)^{\alpha-1} - \delta]\}c(t) + \theta(t)\dot{c}(t)$$

$$0 = \{\rho - [\alpha k(t)^{\alpha-1} - \delta]\}c(t) + \dot{c}(t)$$

$$\dot{c}(t) = [\alpha k(t)^{\alpha-1} - \delta - \rho]c(t)$$

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Now we have a two-dimensional system

$$\dot{c}(t) = [\alpha k(t)^{\alpha-1} - \delta - \rho]c(t)$$

$$\dot{k}(t) = k(t)^{\alpha} - \delta k(t) - c(t)$$

with properties that can be illustrated in a phase diagram with k on the horizontal axis and c on the vertical axis.

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Temporarily rearrange

$$\dot{c}(t) = [\alpha k(t)^{\alpha-1} - \delta - \rho]c(t)$$

as

$$\frac{\dot{c}(t)}{c(t)} + \rho = \alpha k(t)^{\alpha-1} - \delta$$

The left-hand side is the intertemporal marginal rate of substitution; the right-hand side is the intertemporal marginal rate of transformation.

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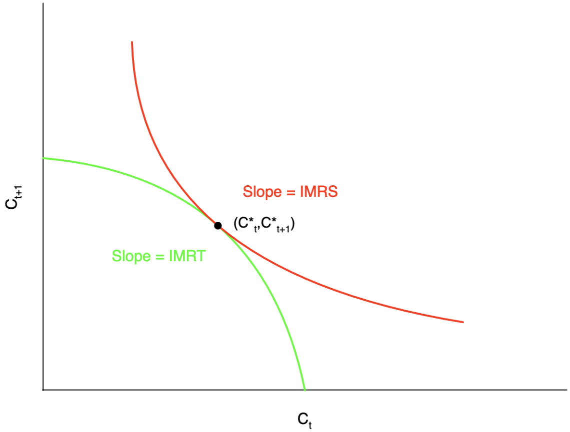

Pareto optimal allocations in the Ramsey model.

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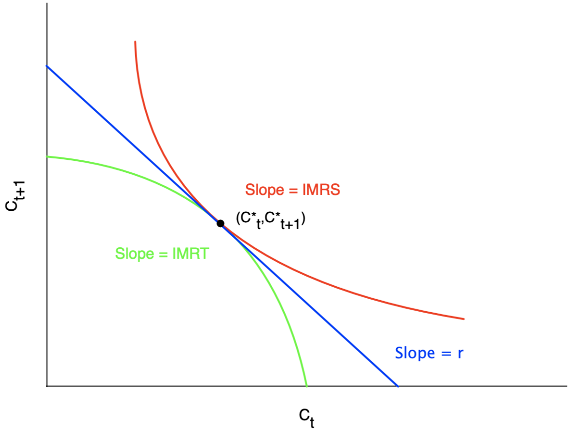

Competitive equilibrium allocations in the Ramsey model.

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$$\dot{c}(t) = [\alpha k(t)^{\alpha-1} - \delta - \rho]c(t)$$

implies

$$\dot{c}(t) = 0 \text{ when } k(t) = k^* = \left(\frac{\delta + \rho}{\alpha}\right)^{1/(\alpha-1)}$$

$$\dot{c}(t) < 0 \text{ when } k(t) > k^*$$

$$\dot{c}(t) > 0 \text{ when } k(t) < k^*$$

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$$\dot{k}(t) = k(t)^\alpha - \delta k(t) - c(t)$$

implies

$$\dot{k}(t) = 0 \text{ when } c(t) = g(k(t)) = k(t)^\alpha - \delta k(t)$$

$$\dot{k}(t) < 0 \text{ when } c(t) > g(k(t))$$

$$\dot{k}(t) > 0 \text{ when } c(t) < g(k(t))$$

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The definition

$$g(k) = k^\alpha - \delta k$$

implies

$$g'(k) = \alpha k^{\alpha-1} - \delta$$

$$g''(k) = (\alpha - 1)\alpha k^{\alpha-2} < 0$$

showing that g is concave.

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In addition

$$g(k) = k^\alpha - \delta k$$

$$g'(k) = \alpha k^{\alpha-1} - \delta$$

imply

$$g'(k^{**}) = 0$$

when

$$k^{**} = \left(\frac{\delta}{\alpha}\right)^{1/(\alpha-1)} > \left(\frac{\delta+\rho}{\alpha}\right)^{1/(\alpha-1)} = k^*$$

g attains its maximum to the right of k^*

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Finally

$$g(k) = k^\alpha - \delta k = [k^{\alpha-1} - \delta]k$$

implies

$$g(k^{***}) = 0$$

when $k^{***} = 0$ or

$$k^{***} = \delta^{1/(\alpha-1)} > \left(\frac{\delta}{\alpha}\right)^{1/(\alpha-1)} = k^{**}$$

g crosses the horizontal axis at $k = 0$ and at the point k^{***} to the right of k^{**}

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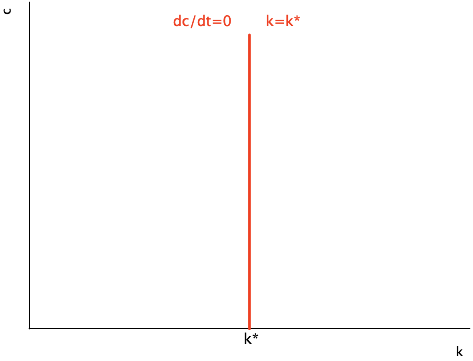

$$\dot{c}(t) = 0 \text{ when } k(t) = k^* = \left(\frac{\delta + \rho}{\alpha}\right)^{1/(\alpha-1)}$$

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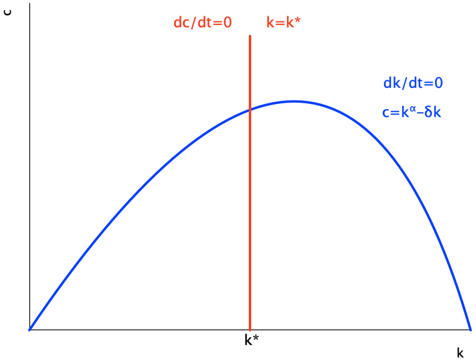

$$\dot{k}(t) = 0 \text{ when } c(t) = g(k(t)) = k(t)^{\alpha} - \delta k(t)$$

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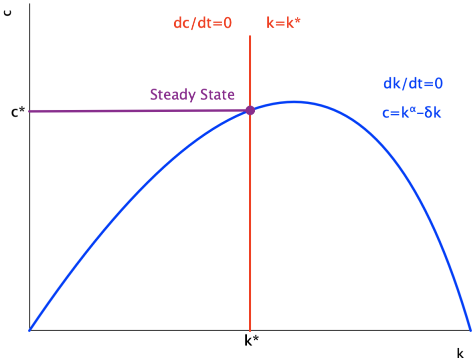

(k^*, c^*) is the unique, nontrivial steady state

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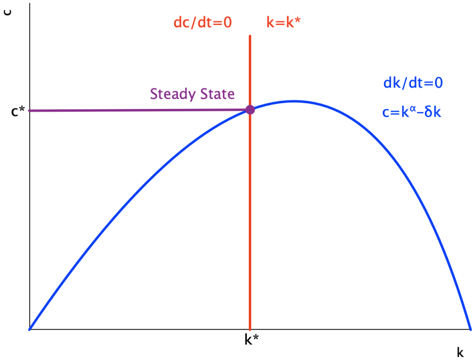

If the economy starts at the steady state, it stays there forever. But suppose the economy starts away from the steady state – does it display any tendency to converge?

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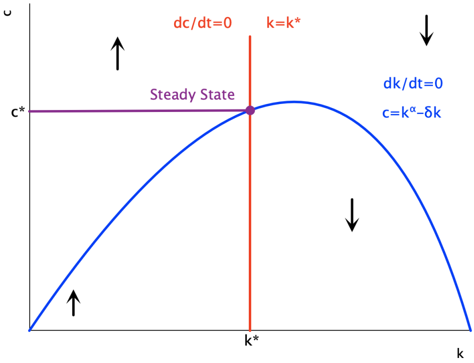

$\dot{c}(t) < 0$ when $k(t) > k^*$ and $\dot{c}(t) > 0$ when $k(t) < k^*$

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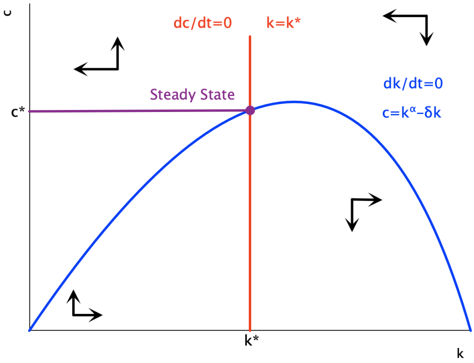

$\dot{k}(t) < 0$ when $c > g(k)$ and $\dot{k}(t) > 0$ when $c < g(k)$

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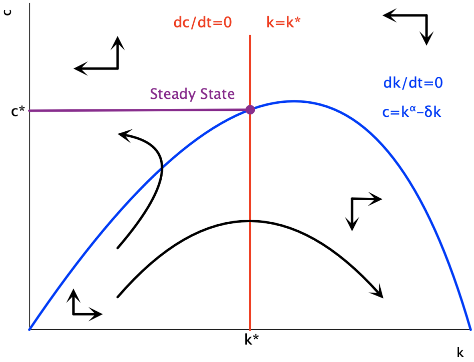

When $k(0) < k^*$, choosing $c(0)$ too high leads to negative capital in finite time and choosing $c(0)$ too low violates the TVC.

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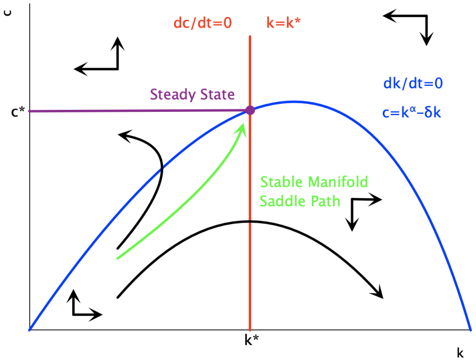

When $k(0) < k^*$, a unique $c(0)$ puts the economy on the stable manifold, converging to the steady state.

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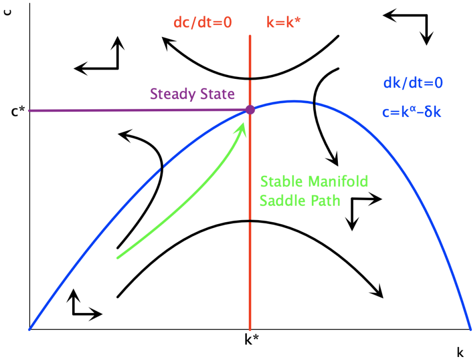

Likewise, when $k(0) > k^*$, choosing $c(0)$ too high leads to negative capital in finite time and choosing $c(0)$ too low violates the TVC.

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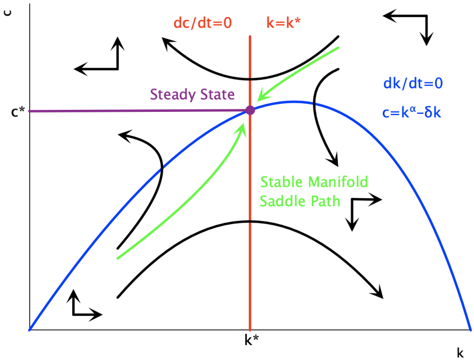

Once again, a unique $c(0)$ puts the economy on the stable manifold, converging to the steady state.

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To see that paths starting from a value of $c(0)$ below the stable manifold violate the TVC, note that the FOC implies

$$\lim_{T \to \infty} e^{-\rho T} \theta(T) k(T) = \lim_{T \to \infty} \frac{e^{-\rho T}}{c(T)} k(T)$$

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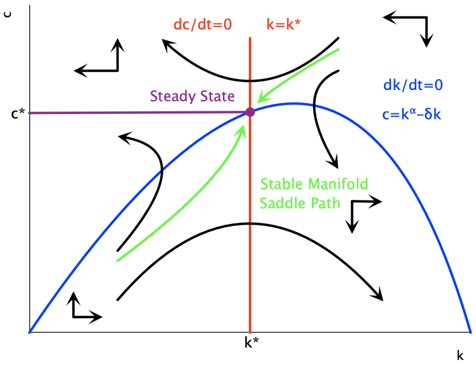

Paths starting from a value of $c(0)$ below the stable manifold have

$$\lim_{T\to\infty} k(T) = k^{***} = \delta^{1/(\alpha-1)} \text{ and } \lim_{T\to\infty} c(T) = 0$$

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Therefore, along paths starting from a value of $c(0)$ below the stable manifold

$$\lim_{T\to\infty} e^{-\rho T}\theta(T)k(T) = \lim_{T\to\infty} \frac{e^{-\rho T}}{c(T)}k(T) = \lim_{T\to\infty} \frac{e^{-\rho T}}{c(T)}k^{***}$$

The TVC is satisfied if $c(T)$ converges to zero at a rate that is slower than ρ, but is violated if $c(T)$ converges to zero at a rate that is faster than ρ.

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$$\dot{c}(t) = [\alpha k(t)^{\alpha-1} - \delta - \rho]c(t)$$

and

$$\lim_{T\to\infty} k(T) = k^{***} = \delta^{1/(\alpha-1)}$$

imply that along paths starting from a value of $c(0)$ below the stable manifold,

$$\begin{aligned}
\lim_{T\to\infty} \frac{\dot{c}(T)}{c(T)} &= \lim_{T\to\infty} [\alpha k(T)^{\alpha-1} - \delta - \rho] \\
&= \alpha\delta - \delta - \rho \\
&= (\alpha - 1)\delta - \rho \\
&< -\rho
\end{aligned}$$

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Along paths starting from a value of $c(0)$ below the stable manifold

$$\lim_{T\to\infty} e^{-\rho T}\theta(T)k(T) = \lim_{T\to\infty} \frac{e^{-\rho T}}{c(T)}k(T) = \lim_{T\to\infty} \frac{e^{-\rho T}}{c(T)}k^{***}$$

and

$$\lim_{T\to\infty} \frac{\dot{c}(T)}{c(T)} < -\rho$$

imply

$$\lim_{T\to\infty} e^{-\rho T}\theta(T)k(T) = \infty$$

The TVC is violated.