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MATH FOR ECONOMISTS

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Optimal Growth

The planner's problem: Given $k(0)$, choose $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ to maximize

$$\int_0^\infty e^{-\rho t} \ln(c(t)) dt$$

subject to

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t) \text{ for all } t \in [0, \infty)$$

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We've already characterized the solution to the planner's problem using the method of Lagrange multipliers.

An alternative is to use the calculus of variations. The general problem takes the form:

$$\max_{y(t),t\in[0,\infty)} \int_0^{\infty} f(y(t), \dot{y}(t), t)dt$$

taking $y(0)$ as given.

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For the Ramsey model, substitute the constraint

$$c(t) = k(t)^{\alpha} - \delta k(t) - \dot{k}(t)$$

into the utility function to restate the problem

$$\max_{k(t), t \in [0, \infty)} \int_0^{\infty} e^{-\rho t} \ln[k(t)^{\alpha} - \delta k(t) - \dot{k}(t)] dt$$

taking $k(0)$ as given.

Optimal Growth

For the Ramsey model

$$\max_{k(t), t \in [0,\infty)} \int_0^\infty e^{-\rho t} \ln[k(t)^\alpha - \delta k(t) - \dot{k}(t)]dt$$

taking $k(0)$ as given.

Use a variational argument and integration by parts to obtain the differential equations (Euler-Lagrange equations) that characterize the solution. See the notes on the maximum principle for details.

The maximum principle lets us derive the same optimality conditions without explicit use of variational arguments and without integration by parts.

Optimal Growth

References:

Dixit, Ch 10

Acemoglu, Ch 7

L.S. Pontryagin et al. *The Mathematical Theory of Optimal Processes* (1962)

Optimal Growth

Consider "restarting" the problem at $t = s \geq 0$: Given $k(s)$, choose $c(t)$ for $t \in [s, \infty)$ and $k(t)$ for $t \in (s, \infty)$ to maximize

$$\int_s^\infty e^{-\rho(t-s)} \ln(c(t)) dt$$

subject to

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t) \text{ for all } t \in [s, \infty)$$

The mathematical structure of the problem is *identical*: $k(0)$ has simply been replaced by $k(s)$.

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Notice that the infinite horizon makes this trick easier to apply. The maximum principle can also be applied to finite horizon problems, but the approach becomes even more compelling in the case of an infinite horizon.

This insight suggests that we can break the problem down, period-by-period, into a a sequence of static problems: at each date t, take $k(t)$ as given and choose $c(t)$, taking into account the effect of this choice on $\ln(c(t))$ *and* $\dot{k}(t)$.

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With this goal in mind, define

$$H(k(t), \theta(t)) = \max_{c(t)} \ln(c(t)) + \theta(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

Using $\theta(t)$ as the "exchange rate" between capital at t and utility at t, both in "current value" (undiscounted) terms.

Although we must still confirm this, our intuition from previous applications of the Kuhn-Tucker and envelope theorems suggests that this will work.

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In

$$H(k(t), \theta(t)) = \max_{c(t)} \ln(c(t)) + \theta(t)[k(t)^\alpha - \delta k(t) - c(t)]$$

$$\tilde{H}(c(t), k(t), \theta(t)) = \ln(c(t)) + \theta(t)[k(t)^\alpha - \delta k(t) - c(t)]$$

is the "current value Hamiltonian."

Choosing $c(t)$ to maximize the CV Hamiltonian is a static, unconstrained optimization problem with a single choice variable.

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In

$$H(k(t), \theta(t)) = \max_{c(t)} \ \ln(c(t)) + \theta(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$
$$= \max_{c(t)} \tilde{H}(c(t), k(t), \theta(t))$$

$H(k(t), \theta(t))$, is the "maximized current value Hamiltonian."

$H(k(t), \theta(t))$ is the maximum value function for the static, unconstrained problem: choose $c(t)$ to maximize the CV Hamiltonian.

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Notice that this approach requires the utility function to be additively time-separable and the constraints to appear in recursive form.

The maximum principle exploits (or depends on) these features.

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$$H(k(t), \theta(t)) = \max_{c(t)} \ \ln(c(t)) + \theta(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

The envelope theorem implies

$$H_k(k(t), \theta(t)) = \theta(t)[\alpha k(t)^{\alpha-1} - \delta]$$

$$H_\theta(k(t), \theta(t)) = k(t)^{\alpha} - \delta k(t) - c(t)$$

where $c(t)$ satisfies the FOC

$$\frac{1}{c(t)} - \theta(t) = 0$$

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The "FOC" for $c(t)$ from the dynamic problem

$$\frac{1}{c(t)} - \theta(t) = 0$$

coincides with the FOC for $c(t)$ from the static problem

$$H(k(t), \theta(t)) = \max_{c(t)} \ \ln(c(t)) + \theta(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

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The "FOC" for $k(t)$ from the dynamic problem

$$\dot{\theta}(t) = \rho\theta(t) - \theta(t)[\alpha k(t)^{\alpha-1} - \delta]$$

can be recovered by applying the envelope theorem to the static problem

$$H(k(t), \theta(t)) = \max_{c(t)} \ln(c(t)) + \theta(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

using

$$\dot{\theta}(t) = \rho\theta(t) - H_k(k(t), \theta(t)).$$

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The constraint from the dynamic problem

$$\dot{k}(t) = k(t)^{\alpha} - \delta k(t) - c(t)$$

can be recovered by applying the envelope theorem to the static problem

$$H(k(t), \theta(t)) = \max_{c(t)} \ln(c(t)) + \theta(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

using

$$\dot{k}(t) = H_{\theta}(k(t), \theta(t))$$

Optimal Growth

Theorem (Maximum Principle) Consider the continuous time dynamic optimization problem: Given $k(0)$, choose $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ to maximize

$$\int_0^\infty e^{-\rho t} \ln(c(t)) dt$$

subject to

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t) \text{ for all } t \in [0, \infty)$$

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Associated with this problem, define the CV Hamiltonian

$$\tilde{H}(c(t), k(t), \theta(t)) = \ln(c(t)) + \theta(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

and the maximized CV Hamiltonian

$$H(k(t), \theta(t)) = \max_{c(t)} \ \tilde{H}(c(t), k(t), \theta(t))$$

$$\max_{c(t)} \ \ln(c(t)) + \theta(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

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Then the solution to the dynamic problem must satisfy: the FOC for the static problem

$$H(k(t), \theta(t)) = \max_{c(t)} \ \ln(c(t)) + \theta(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

$$\frac{1}{c(t)} - \theta(t) = 0$$

for all $t \in [0, \infty)$.

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The pair of differential equations

$$\dot{\theta}(t) = \rho\theta(t) - H_k(k(t), \theta(t))$$

$$\dot{k}(t) = H_\theta(k(t), \theta(t))$$

where the envelope theorem implies

$$\dot{\theta}(t) = \rho\theta(t) - \theta(t)[\alpha k(t)^{\alpha-1} - \delta]$$

$$\dot{k}(t) = k(t)^\alpha - \delta k(t) - c(t)$$

Optimal Growth

Note: The pair of differential equations

$$\dot{\theta}(t) = \rho\theta(t) - H_k(k(t), \theta(t))$$

$$\dot{k}(t) = H_\theta(k(t), \theta(t))$$

takes a form similar to the system

$$\dot{p} = -\frac{\partial H(q, p)}{\partial q}$$

$$\dot{q} = \frac{\partial H(q, p)}{\partial p}$$

derived by physicist William Rowan Hamilton in the 1800s.

Optimal Growth

The initial condition

$$k(0) \text{ given}$$

The transversality condition

$$\lim_{T \to \infty} e^{-\rho T} \theta(T) k(T) = 0$$

See Takashi Kamihigashi, "Necessity of Transversality Conditions for Infinite Horizon Problems," *Econometrica* (2001).

Optimal Growth

Unlike linear differential equations, describing exponential growth or decay, the nonlinear system describing the solution to the Ramsey model lacks a closed-form solution.

But we can deduce the qualitative properties of the solution using a phase diagram, or solve the system numerically.