ECON 772001
MATH FOR ECONOMISTS

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Optimal Growth

The planner's problem: Given $k(0)$, choose $c(t)$ for $t \in [0, T]$ and $k(t)$ for $t \in (0, T]$ to maximize

$$\int_0^T e^{-\rho t} \ln(c(t)) dt$$

subject to

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t) \text{ for all } t \in [0, T]$$

$$k(T) \geq 0$$

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Set up the Lagrangian

$$L(c(t), k(t), \lambda(t), \phi) = \int_0^T e^{-\rho t} \ln(c(t)) dt + \int_0^T \lambda(t)[k(t)^\alpha - \delta k(t) - c(t) - \dot{k}(t)] dt + \phi k(T)$$

Note: Utility is expressed as a "present value" at $t = 0$, while $k(t)$ and $c(t)$ are goods at $t > 0$.

Therefore, $\lambda(t)$ measures the present value at $t = 0$ of capital at $t > 0$.

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Use $\lambda(t) = e^{-\rho t}\theta(t)$ to rewrite the Lagrangian in "current value" form:

$$
\begin{aligned}
L(c(t), k(t), \theta(t), \phi) = &\int_0^T e^{-\rho t} \ln(c(t))dt \\
&+ \int_0^T e^{-\rho t}\theta(t)[k(t)^\alpha - \delta k(t) - c(t)]dt \\
&- \int_0^T e^{-\rho t}\theta(t)\dot{k}(t)dt \\
&+ \phi k(T)
\end{aligned}
$$

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$$
\begin{aligned}
L(c(t), k(t), \theta(t), \phi) = & \int_0^T e^{-\rho t} \ln(c(t)) dt \\
& + \int_0^T e^{-\rho t} \theta(t) [k(t)^\alpha - \delta k(t) - c(t)] dt \\
& - \int_0^T e^{-\rho t} \theta(t) \dot{k}(t) dt \\
& + \phi k(T)
\end{aligned}
$$

How to differentiate the function L with respect to the *functions* $k(t)$ and $c(t)$?

How to differentiate the term involving $\dot{k}(t)$ with respect to $k(t)$?

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To answer the first question – that is, to differentiate L with respect to the functions $k(t)$ and $c(t)$ – we can use a *variational argument*.

For details, see

Acemoglu, Ch.7, Sec.1

Wendell H. Fleming and Raymond W. Rishel. *Deterministic and Stochastic Optimal Control* (1975).

Optimal Growth

Consider the problem: choose $y(t)$, $t \in [0, T]$ to maximize

$$F(y) = \int_0^T f(y(t), t)dt$$

Note that changing $y(t)$ on any set of measure zero has no effect on F. Assume, partly for this reason, that

- y is continuously differentiable
- f is twice continuously differentiable

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Let $y^*(t)$ solve the problem and consider the *admissible variation*

$$y(t) = y^*(t) + \varepsilon\eta(t)$$

where $\varepsilon \in \mathbb{R}$ and η is continuously differentiable.

Evaluate

$$F(y^* + \varepsilon\eta) = \int_0^T f(y^*(t) + \varepsilon\eta(t), t)dt$$

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Since y^* solves the problem, it should not be possible, starting from $\varepsilon = 0$, to choose a small but non-zero value of ε and increase the objective function:

$$\left.\frac{dF(y^* + \varepsilon\eta)}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

$$\left.\frac{d}{d\varepsilon}\int_0^T f(y^*(t) + \varepsilon\eta(t), t)dt\right|_{\varepsilon=0} = 0$$

Since f is (twice) continuously differentiable, we can switch the order of differentiation with respect to ε and integration with respect to t.

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$$\frac{d}{d\varepsilon}\int_0^T f(y^*(t)+\varepsilon\eta(t),t)dt\bigg|_{\varepsilon=0}=0$$

$$\int_0^T f_y(y^*(t)+\varepsilon\eta(t),t)\eta(t)dt\bigg|_{\varepsilon=0}=0$$

$$\int_0^T f_y(y^*(t),t)\eta(t)dt=0$$

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$$\int_0^T f_y(y^*(t), t)\eta(t)dt = 0$$

Since this condition must hold *for any* continuously differentiable function $\eta(t)$, we can choose

$$\eta(t) = f_y(y^*(t), t)$$

to get

$$\int_0^T [f_y(y^*(t), t)]^2 dt = 0$$

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But

$$\int_0^T [f_y(y^*(t), t)]^2 dt = 0$$

requires that

$$f_y(y^*(t), t) = 0$$

for all $t \in [0, T]$.

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Therefore, necessary conditions for a solution to

$$\max_{y(t), t\in[0,T]} \int_0^T f(y(t),t)dt$$

are

$$f_y(y^*(t),t) = 0$$

for all $t \in [0, T]$.

We can fix a value of $t \in [0, T]$ and pretend that $y(t)$ is a variable, not a function.

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Next question: how to differentiate the term

$$\int_0^T e^{-\rho t}\theta(t)\dot{k}(t)dt$$

with respect to $k(t)$?

Answer: rewrite the term in a more convenient form using integration by parts.

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$$\frac{de^{-\rho t}\theta(t)k(t)}{dt} = -\rho e^{-\rho t}\theta(t)k(t) + e^{-\rho t}\dot{\theta}(t)k(t) + e^{-\rho t}\theta(t)\dot{k}(t)$$

$$\int_0^T \frac{de^{-\rho t}\theta(t)k(t)}{dt}dt$$
$$= \int_0^T e^{-\rho t}[\dot{\theta}(t) - \rho\theta(t)]k(t)dt + \int_0^T e^{-\rho t}\theta(t)\dot{k}(t)dt$$

The term on the left can be evaluated using the fundamental theorem of calculus; the term on the right is the one that's giving us trouble in L.

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$$\int_0^T \frac{de^{-\rho t}\theta(t)k(t)}{dt}dt$$
$$= \int_0^T e^{-\rho t}[\dot{\theta}(t) - \rho\theta(t)]k(t)dt + \int_0^T e^{-\rho t}\theta(t)\dot{k}(t)dt$$

$$- \int_0^T e^{-\rho t}\theta(t)\dot{k}(t)dt$$
$$= \int_0^T e^{-\rho t}[\dot{\theta}(t) - \rho\theta(t)]k(t)dt - e^{-\rho T}\theta(T)k(T) + \theta(0)k(0)$$

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Use this result to rewrite the Lagrangian as

$$\begin{aligned} L(c(t), k(t), \theta(t), \phi) = &\int_0^T e^{-\rho t} \ln(c(t)) dt \\ &+ \int_0^T e^{-\rho t} \theta(t)[k(t)^\alpha - \delta k(t) - c(t)] dt \\ &+ \int_0^T e^{-\rho t} [\dot{\theta}(t) - \rho \theta(t)] k(t) dt \\ &- e^{-\rho T} \theta(T) k(T) + \theta(0) k(0) + \phi k(T) \end{aligned}$$

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The maximum principle lets us skip all of these preliminary steps.

But since we've already taken the trouble to work through then, let's see where they lead us.

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$$
\begin{aligned}
L(c(t), k(t), \theta(t), \phi) = & \int_0^T e^{-\rho t} \ln(c(t)) dt + \int_0^T e^{-\rho t} \theta(t)[k(t)^\alpha - \delta k(t) - c(t)] dt \\
& + \int_0^T e^{-\rho t} [\dot{\theta}(t) - \rho \theta(t)] k(t) dt \\
& - e^{-\rho T} \theta(T) k(T) + \theta(0) k(0) + \phi k(T)
\end{aligned}
$$

Fix $t \in [0, T]$. "FOC" for $c(t)$:

$$\frac{e^{-\rho t}}{c(t)} - e^{\rho t} \theta(t) = 0$$

$$\frac{1}{c(t)} - \theta(t) = 0 \text{ for all } t \in [0, T]$$

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$$L(c(t), k(t), \theta(t), \phi) = \int_0^T e^{-\rho t} \ln(c(t))dt + \int_0^T e^{-\rho t}\theta(t)[k(t)^{\alpha} - \delta k(t) - c(t)]dt$$
$$+ \int_0^T e^{-\rho t}[\dot{\theta}(t) - \rho\theta(t)]k(t)dt$$
$$- e^{-\rho T}\theta(T)k(T) + \theta(0)k(0) + \phi k(T)$$

Fix $t \in (0, T)$. "FOC" for $k(t)$:

$$e^{-\rho t}\theta(t)[\alpha k(t)^{\alpha-1} - \delta] + e^{-\rho t}[\dot{\theta}(t) - \rho\theta(t)] = 0$$

$$\dot{\theta}(t) = \rho\theta(t) - \theta(t)[\alpha k(t)^{\alpha-1} - \delta] \text{ for all } t \in [0, T]$$

if all functions of t are continuously differentiable.

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$$L(c(t), k(t), \theta(t), \phi) = \int_0^T e^{-\rho t} \ln(c(t))dt + \int_0^T e^{-\rho t}\theta(t)[k(t)^\alpha - \delta k(t) - c(t)]dt$$
$$+ \int_0^T e^{-\rho t}[\dot{\theta}(t) - \rho\theta(t)]k(t)dt$$
$$- e^{-\rho T}\theta(T)k(T) + \theta(0)k(0) + \phi k(T)$$

"FOC" for $k(T)$:

$$e^{-\rho T}\theta(T)[\alpha k(T)^{\alpha-1} - \delta] + e^{-\rho T}[\dot{\theta}(T) - \rho\theta(T)]$$
$$- e^{-\rho T}\theta(T) + \phi = 0$$

$$\phi = e^{-\rho T}\theta(T)$$

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Binding constraint

$$\dot{k}(t) = k(t)^{\alpha} - \delta k(t) - c(t) \text{ for all } t \in [0, T]$$

Initial condition $k(0)$ given.

Complementary slackness condition

$$\phi k(T) = 0$$

$$e^{-\rho T}\theta(T)k(T) = 0$$

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Consider the simple (linear) differential equation

$$\dot{x}(t) = rx(t)$$

The general solution is

$$x(t) = ke^{rt}$$

Verify by differentiation:

$$\dot{x}(t) = rke^{rt} = rx(t)$$

Optimal Growth

Consider the simple (linear) differential equation

$$\dot{x}(t) = rx(t)$$

The general solution is

$$x(t) = ke^{rt}$$

A specific solution is pinned down by the initial condition $x(0) = x_0$:

$$x_0 = x(0) = ke^0 = k$$

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The *unique* solution to the initial value problem

$$\dot{x}(t) = rx(t), \; x(0) = x_0$$

is

$$x(t) = x_0 e^{rt}$$

Similarly, a system of two differential equations in two unknown functions $k(t)$ and $\theta(t)$ requires two boundary conditions.

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For the Ramsey model, we have an initial condition

$$k(0) \text{ given}$$

and a terminal, or transversality, condition

$$e^{-\rho T}\theta(T)k(T) = 0$$

or, with an infinite horizon,

$$\lim_{T\to\infty} e^{-\rho T}\theta(T)k(T) = 0$$

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With $T < \infty$,

$$e^{-\rho T}\theta(T) = \frac{e^{-\rho T}}{c(T)} > 0$$

Hence, the TVC

$$e^{-\rho T}\theta(T)k(T) = 0$$

requires $k(T) = 0$, preventing overaccumulation.

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With $T = \infty$, it is *not* the case that

$$\lim_{T \to \infty} k(T) = 0$$

Nevertheless, the TVC

$$\lim_{T \to \infty} e^{-\rho T} \theta(T) k(T) = 0$$

prevents overaccumulation.

Optimal Growth

Given $k(0)$, let $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ maximize

$$\int_0^\infty e^{-\rho t} \ln(c(t)) dt$$

subject to

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t) \text{ for all } t \in [0, \infty)$$

Then $c(t)$, $k(t)$, and the associated values of $\theta(t)$ must satisfy:

Optimal Growth

The FOC:

$$\frac{1}{c(t)} - \theta(t) = 0 \text{ for all } t \in [0, \infty)$$

The pair of differential equations:

$$\dot{\theta}(t) = \rho\theta(t) - \theta(t)[\alpha k(t)^{\alpha-1} - \delta] \text{ for all } t \in [0, \infty)$$

$$\dot{k}(t) = k(t)^{\alpha} - \delta k(t) - c(t) \text{ for all } t \in [0, \infty)$$

The initial and transversality conditions:

$$k(0) \text{ given}$$

$$\lim_{T \to \infty} e^{-\rho T}\theta(T)k(T) = 0$$