ECON 772001
MATH FOR ECONOMISTS

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Optimal Growth

The Ramsey Model

Lagrange Multipliers

The Maximum Principle

Phase Diagram

Optimal Growth

Frank Ramsey. "A Mathematical Theory of Saving." *Economic Journal* (1928).

See also: "A Contribution to the Theory of Taxation." *Economic Journal* (1927).

Extended the Solow (*QJE* 1956) model to endogenize the savings rate

David Cass. "Optimal Growth in an Aggregate Model of Capital Accumulation." *Review of Economic Studies* (1965).

Tjalling Koopmans. "On the Concept of Optimal Economic Growth." (1965)

Optimal Growth

Output produced with capital according to

$$y_t = k_t^{\alpha}, 0 < \alpha < 1$$

special case of $y_t = k_t^{\alpha} n_t^{1-\alpha}$, where $n_t = 1$

$\delta =$ depreciation rate for capital

$c_t =$ consumption

$k_{t+1} =$ capital stock at the end of t/beginning of $t + 1$

Optimal Growth

$$k_{t+1} = k_t + k_t^{\alpha} - \delta k_t - c_t$$

$$y_t = k_t^{\alpha} = k_{t+1} - k_t + \delta k_t + c_t = i_t + c_t$$

Government spending and net exports are zero.

Allowing for free disposal:

$$k_t^{\alpha} - \delta k_t - c_t \geq k_{t+1} - k_t$$

Optimal Growth

In discrete time, $\Delta t = 1$:

$$k_t^\alpha - \delta k_t - c_t \geq k_{t+1} - k_t$$

$$[k(t)^\alpha - \delta k(t) - c(t)]\Delta t \geq k(t + \Delta t) - k(t)$$

$$k(t)^\alpha - \delta k(t) - c(t) \geq \frac{k(t + \Delta t) - k(t)}{\Delta t}$$

Optimal Growth

In discrete time, $\Delta t = 1$:

$$k(t)^{\alpha} - \delta k(t) - c(t) \geq \frac{k(t + \Delta t) - k(t)}{\Delta t}$$

In continuous time, $\Delta t \to 0$:

$$k(t)^{\alpha} - \delta k(t) - c(t) \geq \lim_{\Delta t \to 0} \frac{k(t + \Delta t) - k(t)}{\Delta t}$$

$$k(t)^{\alpha} - \delta k(t) - c(t) \geq \dot{k}(t)$$

where

$$\dot{k}(t) = \frac{dk(t)}{dt}$$

Optimal Growth

In continuous time,

$$k(t)^{\alpha} - \delta k(t) - c(t) \geq \dot{k}(t) \text{ for all } t \in [0, T]$$

$$k(0) \text{ given}$$

$$k(T) \geq 0$$

Actually, $k(t) \geq 0$ for all $t \in [0, T]$, but this constraint will only bind at T.

Optimal Growth

Utility

$$\int_0^T e^{-\rho t} \ln(c(t))dt$$

$\rho > 0$ discount rate

In this economy, the welfare theorems hold:

1) Allocations in any competitive equilibrium are Pareto optimal.

2) Any Pareto optimal allocation can be supported in a competitive equilibrium.

Optimal Growth

The planner's problem: Given $k(0)$, choose $c(t)$ for $t \in [0, T]$ and $k(t)$ for $t \in (0, T]$ to maximize

$$\int_0^T e^{-\rho t} \ln(c(t)) dt$$

subject to

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t) \text{ for all } t \in [0, T]$$

$$k(T) \geq 0$$

Optimal Growth

Set up the Lagrangian

$$
\begin{aligned}
L(c(t), k(t), \lambda(t), \phi) = &\int_0^T e^{-\rho t} \ln(c(t)) dt \\
&+ \int_0^T \lambda(t)[k(t)^\alpha - \delta k(t) - c(t) - \dot{k}(t)] dt \\
&+ \phi k(T)
\end{aligned}
$$

Note: Utility is expressed as a "present value" at $t = 0$, while $k(t)$ and $c(t)$ are goods at $t > 0$.

Therefore, $\lambda(t)$ measures the present value at $t = 0$ of capital at $t > 0$.

Optimal Growth

Set up the Lagrangian

$$
\begin{aligned}
L(c(t), k(t), \lambda(t), \phi) = &\int_0^T e^{-\rho t} \ln(c(t)) dt \\
&+ \int_0^T \lambda(t)[k(t)^\alpha - \delta k(t) - c(t) - \dot{k}(t)] dt \\
&+ \phi k(T)
\end{aligned}
$$

$\lambda(t)$ measures the present value at $t = 0$ of capital at $t > 0$.

Define $\theta(t) = e^{\rho t}\lambda(t)$, so that $\lambda(t) = e^{-\rho t}\theta(t)$.

Then $\theta(t)$ measures the "current value" at $t > 0$ of capital at $t > 0$.

Optimal Growth

Use $\lambda(t) = e^{-\rho t}\theta(t)$ to rewrite the Lagrangian in "current value" form:

$$
\begin{aligned}
L(c(t), k(t), \theta(t), \phi) = & \int_0^T e^{-\rho t} \ln(c(t)) dt \\
& + \int_0^T e^{-\rho t}\theta(t)[k(t)^\alpha - \delta k(t) - c(t)] dt \\
& - \int_0^T e^{-\rho t}\theta(t)\dot{k}(t) dt \\
& + \phi k(T)
\end{aligned}
$$

Optimal Growth

$$\begin{aligned} L(c(t), k(t), \theta(t), \phi) = & \int_0^T e^{-\rho t} \ln(c(t)) dt \\ & + \int_0^T e^{-\rho t} \theta(t)[k(t)^\alpha - \delta k(t) - c(t)] dt \\ & - \int_0^T e^{-\rho t} \theta(t) \dot{k}(t) dt \\ & + \phi k(T) \end{aligned}$$

How to differentiate the function L with respect to the *functions* $k(t)$ and $c(t)$?

How to differentiate the term involving $\dot{k}(t)$ with respect to $k(t)$?