ECON 772001
MATH FOR ECONOMISTS

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Generalizing the Results

To generalize our proof of the Kuhn-Tucker theorem, we will make repeated use of the implicit function theorem.

For details, see

Simon and Blume, Chapter 15

Acemoglu, Appendix A

The version we will need is not as general.

Generalizing the Results

Consider a system of n equations involving n variables $y_1, y_2, \ldots, y_n$ and n parameters $c_1, c_2, \ldots, c_n$:

$$\begin{aligned} H_1(y_1, y_2, \ldots, y_n) &= c_1 \\ H_2(y_1, y_2, \ldots, y_n) &= c_2 \\ &\vdots \\ H_n(y_1, y_2, \ldots, y_n) &= c_n \end{aligned}$$

Note: there can be more than n parameters, and each parameter can enter more than one equation, nonlinearly. But there must be at least n variables.

Generalizing the Results

Suppose that for a given set of parameters $c_1^*, c_2^*, \ldots, c_n^*$, all of the equations are satisfied at $y_1^*, y_2^*, \ldots, y_n^*$:

$$
\begin{aligned}
H_1(y_1^*, y_2^*, \ldots, y_n^*) &= c_1^* \\
H_2(y_1^*, y_2^*, \ldots, y_n^*) &= c_2^* \\
&\vdots \\
H_n(y_1^*, y_2^*, \ldots, y_n^*) &= c_n^*
\end{aligned}
$$

The question is: under what conditions will the y's vary smoothly with the c's?

Generalizing the Results

Assume (a) that each H_j, $j = 1, 2, \ldots, n$ is continuously differentiable and that the matrix

$$\begin{bmatrix} \partial H_1/\partial y_1 & \partial H_1/\partial y_2 & \ldots & \partial H_1/\partial y_n \\ \partial H_2/\partial y_1 & \partial H_2/\partial y_2 & \ldots & \partial H_2/\partial y_n \\ \vdots & \vdots & \ldots & \vdots \\ \partial H_n/\partial y_1 & \partial H_n/\partial y_2 & \ldots & \partial H_n/\partial y_n \end{bmatrix}$$

is nonsingular at $y_1^*, y_2^*, \ldots, y_n^*$.

Generalizing the Results

Then there exist continuously differentiable functions $y_1(c_1, c_2, \ldots, c_n), y_2(c_1, c_2, \ldots, c_n), \ldots, y_n(c_1, c_2, \ldots, c_n)$, defined on an open set $C \subseteq \mathbb{R}^n$ containing $c_1^*, c_2^*, \ldots, c_n^*$, such that

$$\begin{aligned} H_1(y_1(c_1, c_2, \ldots, c_n), \ldots, y_n(c_1, c_2, \ldots, c_n)) &= c_1 \\ H_2(y_1(c_1, c_2, \ldots, c_n), \ldots, y_n(c_1, c_2, \ldots, c_n)) &= c_2 \\ &\vdots \\ H_n(y_1(c_1, c_2, \ldots, c_n), \ldots, y_n(c_1, c_2, \ldots, c_n)) &= c_n \end{aligned}$$

for all $(c_1, c_2, \ldots, c_n) \in C$.

Generalizing the Results

With this result in mind, let's generalize our previous problem to include n choice variables and m constraints, where n and m are arbitrarily large but finite:

choice variables $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

objective function $F(x) = F(x_1, x_2, \ldots, x_n)$

$F : \mathbb{R}^n \to \mathbb{R}$ continuously differentiable

constraints $c_j \geq G_j(x) = G(x_1, x_2, \ldots, x_n)$, $j = 1, 2, \ldots, m$

$c_j \in \mathbb{R}$, $G_j : \mathbb{R}^n \to \mathbb{R}$ continuously differentiable

Typically, $m \leq n$, or at least $\bar{m} \leq n$, where $\bar{m}$ is the number of binding constraints.

Generalizing the Results

The problem:

$$\max_{x_1, x_2, \ldots, x_n} F(x_1, x_2, \ldots, x_n) \text{ subject to}$$

$$c_j \geq G(x_1, x_2, \ldots, x_n) \text{ for all } j = 1, 2, \ldots, m$$

The Lagrangian:

$$L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = F(x_1, x_2, \ldots, x_n) + \sum_{j=1}^{m} \lambda_j [c_j - G_j(x_1, x_2, \ldots, x_n)]$$

Generalizing the Results

Theorem (Kuhn-Tucker) Let $x^* = (x_1^*, \ldots, x_n^*)$ maximize $F(x)$ subject to $c_j \geq G_j(x)$ for all $j = 1, 2, \ldots, m$, where F and G_j, $j = 1, 2, \ldots, m$, are all continuously differentiable. Suppose, without loss of generality, that the first $\bar{m}$ constraints, $0 \leq \bar{m} \leq m$, bind at the optimum, while the remaining $m - \bar{m}$ constraints are nonbinding. Suppose, as well, that the matrix

$$\begin{bmatrix} G_{11}(x^*) & G_{12}(x^*) & \ldots & G_{1n}(x^*) \\ G_{21}(x^*) & G_{22}(x^*) & \ldots & G_{2n}(x^*) \\ \vdots & \vdots & \ldots & \vdots \\ G_{\bar{m}1}(x^*) & G_{\bar{m}2}(x^*) & \ldots & G_{\bar{m}n}(x^*) \end{bmatrix} \tag{12}$$

where $G_{ji}(x^*) = \partial G_j(x^*)/\partial x_i$, has maximal rank $\bar{m}$.

Generalizing the Results

Then there exist values $\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*$ that, together with $x_1^*, \ldots, x_n^*$, satisfy the first-order conditions

$$L_i(x_1^*, \ldots, x_n^*, \lambda_1^*, \ldots, \lambda_m^*) = F_i(x_1^*, \ldots, x_n^*) - \sum_{j=1}^{m} \lambda_j^* G_{ji}(x_1^*, \ldots, x_n^*) = 0 \quad (13)$$

for all $i = 1, 2, \ldots, n$

Generalizing the Results

the constraints

$$L_{n+j}(x_1^*, \ldots, x_n^*, \lambda_1^*, \ldots, \lambda_m^*) = c_j - G_j(x_1^*, \ldots, x_n^*) \geq 0 \quad (14)$$

for all $j = 1, 2, \ldots, m$, the nonnegativity conditions

$$\lambda_j^* \geq 0 \quad (15)$$

for all $j = 1, 2, \ldots, m$, and the complementary slackness conditions

$$\lambda_j^*[c_j - G_j(x_1^*, \ldots, x_n^*)] = 0 \quad (16)$$

for all $j = 1, 2, \ldots, m$