ECON 772001
MATH FOR ECONOMISTS

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Cost Minimization Example

Apply the Kuhn-Tucker theorem to a constrained minimization problem instead of a constrained maximization problem,

Use the envelope theorem to obtain Shephard's lemma and a version of Le Chatelier's principle.

Cost Minimization Example

A firm produces output y with capital k and labor l subject to

$$f(k, l) \geq y$$

$$r = \text{ rental rate for capital}$$

$$w = \text{ wage rate for labor}$$

Suppose the firm minimizes the cost of producing output y:

$$\min_{k,l} rk + wl \text{ subject to } f(k, l) \geq y$$

Cost Minimization Example

Define the Lagrangian by subtracting rather than adding the term involving the multiplier and constraint:

$$L(k, l, \lambda) = rk + wl - \lambda[f(k, l) - y]$$

FOCs:

$$r - \lambda^* f_1(k^*, l^*) = 0 \tag{9}$$

$$w - \lambda^* f_2(k^*, l^*) = 0 \tag{10}$$

If we have no special interest in the value of λ^*, we could combine the FOCs to obtain

$$\frac{r}{w} = \frac{f_1(k^*, l^*)}{f_2(k^*, l^*)}$$

Cost Minimization Example

Alternatively, use the FOCs together with the binding constraint

$$r - \lambda^* f_1(k^*, l^*) = 0 \tag{9}$$

$$w - \lambda^* f_2(k^*, l^*) = 0 \tag{10}$$

$$f(k^*, l^*) = y \tag{11}$$

as a system of 3 equations to find the three unknowns: k^*, l^*, and λ^*.

$k^*(r, w, y)$ and $l^*(r, w, y)$ are conditional factor demand curves. We can apply the envelope theorem to interpret $\lambda^*(r, w, y)$.

Cost Minimization Example

Define the minimum cost function

$$\begin{aligned} C(r,w,y) &= \min_{k,l} rk + wl \text{ subject to } f(k,l) \geq y \\ &= rk^*(r,w,y) + wl^*(r,w,y) \\ &= rk^*(r,w,y) + wl^*(r,w,y) \\ &\quad - \lambda^*(r,w,y)\{f[k^*(r,w,y), l^*(r,w,y)] - y\} \end{aligned}$$

Use the envelope theorem to compute

$$C_1(r,w,y) = k^*(r,w,y)$$

$$C_2(r,w,y) = l^*(r,w,y)$$

$$C_3(r,w,y) = \lambda^*(r,w,y)$$

Cost Minimization Example

$$C_1(r, w, y) = k^*(r, w, y)$$
$$C_2(r, w, y) = l^*(r, w, y)$$
$$C_3(r, w, y) = \lambda^*(r, w, y)$$

The first two conditions restate Shephard's lemma: the partial derivative of the minimum cost function with respect to a factor price coincides with the conditional factor demand curve.

The third condition provides an interpretation of $\lambda^*(r, w, y)$ as the marginal cost of y.

Cost Minimization Example

In chemistry, Le Chatelier's principle says that if a chemical equilibrium is disturbed by a change in temperature, pressure, or concentration, then all of these variables will adjust so as to restore the system to a new equilibrium.

In economics, Samuleson used the same term to describe a set of results that say, for example, that the demand for labor will be more responsive to a change in the wage in the long run, when capital can adjust, then in the short run, when capital is fixed.

Cost Minimization Example

With two factors of production and an output requirement, holding k fixed in the short run also means that l can't change. Le Chatelier's principle holds, but does so trivially.

To make the result more interesting, introduce a third factor of production

$$m = \text{ materials input}$$

$$q = \text{ materials price}$$

Cost Minimization Example

Define

$$C(r, w, q, y) = \min_{k,l,m} rk + wl + qm \text{ subject to } f(k, l, m) \geq y$$

The envelope theorem (Shephard's lemma) implies

$$C_1(r, w, q, y) = k^*(r, w, q, y)$$

$$C_2(r, w, q, y) = l^*(r, w, q, y)$$

$$C_3(r, w, q, y) = m^*(r, w, q, y)$$

$$C_4(r, w, q, y) = \lambda^*(r, w, q, y)$$

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If factor demands are continuously differentiable, then the cost function is twice continuously differentiable. The envelope theorem also implies

$$C_1(r, w, q, y) = k^*(r, w, q, y)$$

$$C_2(r, w, q, y) = l^*(r, w, q, y)$$

$$C_{12}(r, w, q, y) = k_2^*(r, w, q, y)$$

$$C_{21}(r, w, q, y) = l_1^*(r, w, q, y)$$

$$\begin{aligned} C_{12}(r, w, q, y) = \ & C_{21}(r, w, q, y) \Rightarrow \\ & k_2^*(r, w, q, y) = l_1^*(r, w, q, y) \end{aligned}$$

Factor demands must satisfy a “reciprocity” condition.

Cost Minimization Example

Now consider a "short-run" version of the problem, with $k = \bar{k}$ fixed:

$$\min_{l,m} r\bar{k} + wl + qm \text{ subject to } f(\bar{k}, l, m) \geq y$$

$$L(l, m, \mu) = r\bar{k} + wl + qm - \mu[f(\bar{k}, l, m) - y]$$

FOCs

$$w - \mu^s f_2(\bar{k}, l^s, m^s) = 0$$

$$q - \mu^s f_3(\bar{k}, l^s, m^s) = 0$$

Binding constraint

$$f(\bar{k}, l^s, m^s) = y$$

Cost Minimization Example

$$w - \mu^s f_2(\bar{k}, l^s, m^s) = 0$$

$$q - \mu^s f_3(\bar{k}, l^s, m^s) = 0$$

$$f(\bar{k}, l^s, m^s) = y$$

Note that these optimality conditions depend on $\bar{k}$ but not r. Hence

$$l^s = l^s(w, q, y, \bar{k}) \text{ and } m^s = l^s(w, q, y, \bar{k})$$

describe short-run conditional factor demands and

$$\mu^s = \mu^s(w, q, y, \bar{k})$$

measures short-run marginal cost.

Cost Minimization Example

$$w - \mu^s f_2(\bar{k}, l^s, m^s) = 0$$
$$q - \mu^s f_3(\bar{k}, l^s, m^s) = 0$$
$$f(\bar{k}, l^s, m^s) = y$$

Note also that these short-run optimality conditions take the same form as the long-run optimality conditions, but with $\bar{k}$ replaced by k^*:

$$w - \lambda^* f_2(k^*, l^*, m^*) = 0$$

$$q - \lambda^* f_3(k^*, l^*, m^*) = 0$$
$$f(k^*, l^*, m^*) = y$$

Cost Minimization Example

Therefore, if $\bar{k} = k^*(r, w, q, y)$, then the short-run and long-run factor demands will coincide. For labor:

$$l^s[w, q, y, k^*(r, w, q, y)] = l^*(r, w, q, y)$$

And since this equality holds for all (r, w, q, y), we can differentiate with respect to w:

$$\begin{aligned} & l_1^s[w, q, y, k^*(r, w, q, y)] \\ & \quad + l_4^s[w, q, y, k^*(r, w, q, y)]k_2^*(r, w, q, y) \\ = \; & l_2^*(r, w, q, y) \end{aligned}$$

Cost Minimization Example

$$\begin{aligned} & l_1^s[w, q, y, k^*(r, w, q, y)] \\ & \quad + l_4^s[w, q, y, k^*(r, w, q, y)] k_2^*(r, w, q, y) \\ = \ & l_2^*(r, w, q, y) \end{aligned}$$

The red term measures the responsiveness of labor demand to a change in the wage in the short run. The green term measures the responsiveness of labor demand to a change in the wage in the long run. The SOCs imply that both are negative. But Le Chatelier's principle states that the long-run response should be "more negative." This requires that the blue term is also negative.

Cost Minimization Example

$$l_4^s[w,q,y,k^*(r,w,q,y)]k_2^*(r,w,q,y)$$

The first term shows how l^s responds to a change in $\bar{k}$.

The second term shows how k^* responds to a change in w.

Whether or not the product is negative would seem to depend, in a complicated way, on whether capital and labor are substitutes or complements in the short run and the long run.

It turns out, however, that we can say something definite.

Cost Minimization Example

To see how, return to

$$l^s[w, q, y, k^*(r, w, q, y)] = l^*(r, w, q, y)$$

and differentiate this time with respect to r:

$$l_4^s[w, q, y, k^*(r, w, q, y)]k_1^*(r, w, q, y) = l_1^*(r, w, q, y)$$

$$l_4^s[w, q, y, k^*(r, w, q, y)] = \frac{l_1^*(r, w, q, y)}{k_1^*(r, w, q, y)}.$$

Cost Minimization Example

Using

$$l_4^s[w, q, y, k^*(r, w, q, y)] = \frac{l_1^*(r, w, q, y)}{k_1^*(r, w, q, y)}$$

$$l_4^s[w, q, y, k^*(r, w, q, y)]k_2^*(r, w, q, y)$$
$$= \frac{l_1^*(r, w, q, y)k_2^*(r, w, q, y)}{k_1^*(r, w, q, y)}$$

The term in the denominator is negative by the SOCs. The two terms in the numerator have the same sign by reciprocity. The blue term is negative!

Cost Minimization Example

Therefore,

$$l_1^s[w,q,y,k^*(r,w,q,y)] \geq l_2^*(r,w,q,y)$$

in general and, so long as $l_1^*(r,w,q,y) = k_2^*(r,w,q,y) \neq 0$

$$l_1^s[w,q,y,k^*(r,w,q,y)] > l_2^*(r,w,q,y)$$

Finally, if $\bar{k}$ is close to k^*:

$$l_1^s(w,q,y,\bar{k}) > l_2^*(r,w,q,y)$$

which is Le Chatelier's principle.

Cost Minimization Example

$$l_1^s(w, q, y, \bar{k}) > l_2^*(r, w, q, y)$$

Note that Le Chatelier's principle is a "local" result in two ways:

The change in w must be small

$\bar{k}$ must be close to k^*

Cost Minimization Example

With this example we've seen how to apply the Kuhn-Tucker and envelope theorems to constrained minimization as well as constrained maximization problems.

We've also seen how many famous results from producer theory are really just special cases of or applications of the envelope theorem.

Next, we'll generalize these results to apply to problems with arbitrarily large (but finite) numbers of choice variables and constraints.