ECON 772001
MATH FOR ECONOMISTS

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September 10, 2020

The Envelope Theorem

Define the maximum value function

$$V(\theta) = \max_{x} F(x, \theta) \text{ subject to } c \geq G(x, \theta)$$

Evaluating $V(\theta)$ requires two steps:

1) Given θ, find x^*

2) Evaluate $V(\theta) = F(x^*, \theta)$

The Envelope Theorem

Given θ, find x^*: by the Kuhn-Tucker theorem:

$$\lambda^*[c - G(x^*, \theta)] = 0$$

Therefore

$$\begin{aligned} V(\theta) &= F(x^*, \theta) \\ &= F(x^*, \theta) + \lambda^*[c - G(x^*, \theta)] \end{aligned}$$

The Envelope Theorem

$$V(\theta) = F(x^*, \theta) + \lambda^*[c - G(x^*, \theta)]$$

Because this expression holds for all θ, differentiate both sides with respect to θ to get

$$V'(\theta) = F_2(x^*, \theta) - \lambda^* G_2(x^*, \theta)$$

The envelope theorem confirms that this result is true even though, in deriving it here, we've ignored the important fact that x^* and λ^* depend on θ.

The Envelope Theorem

Theorem (Envelope) Let F and G be continuously differentiable functions of x and θ. For any given θ, let $x^*(\theta)$ maximize $F(x,\theta)$ subject to $c \geq G(x,\theta)$, and the $\lambda^*(\theta)$ be the corresponding value of the Lagrange multiplier. For all θ, assume that the constraint qualification $G_1[x^*(\theta),\theta] \neq 0$ holds. Assume, as well, that both $x^*(\theta)$ and $\lambda^*(\theta)$ are continuously differentiable functions. Then the maximum value function defined by

$$V(\theta) = \max_x F(x,\theta) \text{ subject to } c \geq G(x,\theta)$$

satisfies

$$V'(\theta) = F_2[x^*(\theta),\theta] - \lambda^*(\theta) G_2[x^*(\theta),\theta]. \tag{7}$$

The Envelope Theorem

Proof By the Kuhn-Tucker theorem, $x^*(\theta)$ and $\lambda^*(\theta)$ satisfy

$$L_1[x^*(\theta), \lambda^*(\theta)] = F_1[x^*(\theta), \theta] - \lambda^*(\theta) G_1[x^*(\theta), \theta] = 0 \quad (1)$$

and

$$\lambda^*(\theta)\{c - G[x^*(\theta), \theta]\} = 0 \quad (4)$$

for all values of θ.

The Envelope Theorem

The definitions of V and $x^*(\theta)$ imply

$$V(\theta) = F[x^*(\theta), \theta]$$

and hence (4) implies

$$V(\theta) = F[x^*(\theta), \theta] + \lambda^*(\theta)\{c - G[x^*(\theta), \theta]\}$$

for all θ.

The Envelope Theorem

Differentiate

$$V(\theta) = F[x^*(\theta), \theta] + \lambda^*(\theta)\{c - G[x^*(\theta), \theta]\}$$

with respect to θ to get

$$\begin{aligned} V'(\theta) = {} & F_1[x^*(\theta), \theta]x^{*\prime}(\theta) + F_2[x^*(\theta), \theta] \\ & + \lambda^{*\prime}(\theta)\{c - G[x^*(\theta), \theta]\} \\ & - \lambda^*(\theta)G_1[x^*(\theta), \theta]x^{*\prime}(\theta) - \lambda^*(\theta)G_2[x^*(\theta), \theta] \end{aligned}$$

The Envelope Theorem

$$\begin{aligned} V'(\theta) = &\; F_1[x^*(\theta), \theta]x^{*\prime}(\theta) + F_2[x^*(\theta), \theta] \\ &+ \lambda^{*\prime}(\theta)\{c - G[x^*(\theta), \theta]\} \\ &-\lambda^*(\theta)G_1[x^*(\theta), \theta]x^{*\prime}(\theta) - \lambda^*(\theta)G_2[x^*(\theta), \theta] \end{aligned}$$

In general, $x^{*\prime}(\theta) \neq 0$: x^* will depend on θ.

However, (1) implies that x^* is a critical point of L: the effect of a small change in x^* on V is negligible.

The Envelope Theorem

$$\begin{aligned} V'(\theta) = &\ F_1[x^*(\theta), \theta]x^{*\prime}(\theta) + F_2[x^*(\theta), \theta] \\ &+ \lambda^{*\prime}(\theta)\{c - G[x^*(\theta), \theta]\} \\ &-\lambda^*(\theta)G_1[x^*(\theta), \theta]x^{*\prime}(\theta) - \lambda^*(\theta)G_2[x^*(\theta), \theta] \end{aligned}$$

$$\begin{aligned} V'(\theta) = &\ F_2[x^*(\theta), \theta] - \lambda^*(\theta)G_2[x^*(\theta), \theta] \\ &+ \lambda^{*\prime}(\theta)\{c - G[x^*(\theta), \theta]\} \end{aligned}$$

It only remains to show that

$$\lambda^{*\prime}(\theta)\{c - G[x^*(\theta), \theta]\} = 0 \tag{8}$$

The Envelope Theorem

Clearly

$$\lambda^{*\prime}(\theta)\{c - G[x^*(\theta), \theta]\} = 0 \tag{8}$$

holds for any value of θ such that

$$c = G[x^*(\theta), \theta]$$

The Envelope Theorem

So suppose instead that θ such that

$$c > G[x^*(\theta), \theta]$$

For this value of θ, (4) requires that $\lambda^*(\theta) = 0$. Moreover, since G and x^* are both continuously differentiable, there exists an $\varepsilon^* > 0$ such that, for all ε satisfying $\varepsilon^* > |\varepsilon|$,

$$c > G[x^*(\theta + \varepsilon), \theta + \varepsilon]$$

and, by (4), $\lambda^*(\theta + \varepsilon) = 0$ as well.

The Envelope Theorem

If θ such that

$$c > G[x^*(\theta), \theta]$$

then $\lambda^*(\theta) = 0$ and $\lambda^*(\theta + \varepsilon) = 0$ as well.

$\lambda^*(\theta)$ is a constant function in a neighborhood of θ:

$$\lambda^{*\prime}(\theta) = \lim_{\varepsilon \to 0} \frac{\lambda^*(\theta + \varepsilon) - \lambda^*(\theta)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{0}{\varepsilon} = 0$$

implying that (8) and (7) hold for this case as well.

The Envelope Theorem

To develop intuition for the envelope theorem, consider the unconstrained problem

$$\max_x F(x, \theta)$$

Define

$$V(\theta) = \max_x F(x, \theta)$$

The Envelope Theorem

As before, for any value of θ, find $x^*(\theta)$ and evaluate

$$V(\theta) = F[x^*(\theta), \theta]$$

then differentiate

$$V'(\theta) = F_1[x^*(\theta), \theta]x^{*\prime}(\theta) + F_2[x^*(\theta), \theta]$$

In general, $x^{*\prime}(\theta) \neq 0$. But

$$F_1[x^*(\theta), \theta] = 0$$

since $x^*(\theta)$ is chosen optimally.

The Envelope Theorem

Therefore

$$V'(\theta) = F_2[x^*(\theta), \theta]$$

which is the envelope theorem for the unconstrained case.

For the constrained case, V must be evaluated as

$$V(\theta) = F[x^*(\theta), \theta] + \lambda^*(\theta)\{c - G[x^*(\theta), \theta]\}$$

since $x^*(\theta)$ is a critical value of L, not F.

The Envelope Theorem

To see where the "envelope" theorem gets its name, see the graphical analysis in the notes or in Dixit's book.

In the notes, there is an unconstrained example where a firm chooses labor input to maximize profits. The envelope theorem in this case leads to a version of Hotelling's lemma, which says that the derivative of the profit function with respect to the wage equals -1 times the labor demand function.

Finally, in the notes, there is also an application a consumer's utility maximization problem that confirms our earlier intuition that λ^* measures the marginal utility of income.