ECON 772001
MATH FOR ECONOMISTS

Peter Ireland

Boston College

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The Envelope Theorem

References:

Dixit, Chapter 5

Simon and Blume, Chapter 19

Acemoglu, Appendix A

The Envelope Theorem

Shephard's Lemma, Hotelling's Lemma, Roy's Identity, and Le Chatelier's Principle are all applications of the Envelope Theorem

Paul Samuelson. *Foundations of Economic Analysis* (1947).

The Envelope Theorem

Now extend the constrained optimization problem by introducing a parameter θ:

$$\max_{x} F(x, \theta) \text{ subject to } c \geq G(x, \theta)$$

Define the maximum value function

$$V(\theta) = \max_{x} F(x, \theta) \text{ subject to } c \geq G(x, \theta)$$

The Envelope Theorem

Define the maximum value function

$$V(\theta) = \max_x F(x, \theta) \text{ subject to } c \geq G(x, \theta)$$

Evaluating $V(\theta)$ requires two steps:

1) Given θ, find x^*

2) Evaluate $V(\theta) = F(x^*, \theta)$

The Envelope Theorem

Given θ, find x^*: by the Kuhn-Tucker theorem:

$$\lambda^*[c - G(x^*, \theta)] = 0$$

Therefore

$$\begin{aligned} V(\theta) &= F(x^*, \theta) \\ &= F(x^*, \theta) + \lambda^*[c - G(x^*, \theta)] \end{aligned}$$

The Envelope Theorem

$$V(\theta) = F(x^*, \theta) + \lambda^*[c - G(x^*, \theta)]$$

Because this expression holds for all θ, differentiate both sides with respect to θ to get

$$V'(\theta) = F_2(x^*, \theta) - \lambda^* G_2(x^*, \theta)$$

The envelope theorem confirms that this result is true even though, in deriving it here, we've ignored the important fact that x^* and λ^* depend on θ.

The Envelope Theorem

Theorem (Envelope) Let F and G be continuously differentiable functions of x and θ. For any given θ, let $x^*(\theta)$ maximize $F(x, \theta)$ subject to $c \geq G(x, \theta)$, and the $\lambda^*(\theta)$ be the corresponding value of the Lagrange multiplier. For all θ, assume that the constraint qualification $G_1[x^*(\theta), \theta] \neq 0$ holds. Assume, as well, that both $x^*(\theta)$ and $\lambda^*(\theta)$ are continuously differentiable functions. Then the maximum value function defined by

$$V(\theta) = \max_x F(x, \theta) \text{ subject to } c \geq G(x, \theta)$$

satisfies

$$V'(\theta) = F_2[x^*(\theta), \theta] - \lambda^*(\theta) G_2[x^*(\theta), \theta]. \tag{7}$$