ECON 772001
MATH FOR ECONOMISTS

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The Kuhn-Tucker and Envelope Theorems

1. The Kuhn-Tucker Theorem
2. The Envelope Theorem
3. Example: Cost Minimization
4. Generalizing the Results
5. Berge's Maximum Theorem

The Kuhn-Tucker Theorem

References:

Dixit, Chapters 2 and 3

Simon and Blume, Chapters 18 and 19

Acemoglu, Appendix A

The Kuhn-Tucker Theorem

History:

Joseph-Louis Lagrange - mid/late 1700s - optimization subject to equality constraints.

Harold Kuhn and Albert Tucker, "Nonlinear Programming" (1951) - inequality constraints and nonnegativitiy constraints on choice variables.

William Karush, MA Thesis (1939).

Richard Cottle. "William Karush and the KKT Theorem." *Documenta Mathematica* (2012): pp.255-269.

The Kuhn-Tucker Theorem

Consider the constrained optimization problem:

$x \in \mathbb{R}$ choice variable

$F : \mathbb{R} \to \mathbb{R}$ objective function, continuously differentiable

$c \geq G(x)$ constraint

$c \in \mathbb{R}$, $G : \mathbb{R} \to \mathbb{R}$ continuously differentiable

The Kuhn-Tucker Theorem

Associated with the problem:

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

Define the Lagrangian $L : \mathbb{R}^2 \to \mathbb{R}$:

$$L(x, \lambda) = F(x) + \lambda[c - G(x)]$$

The Kuhn-Tucker Theorem

Theorem (Kuhn-Tucker) Let x^* maximize $F(x)$ subject to $c \geq G(x)$, where F and G are both continuously differentiable. Assume, as well, that $G'(x^*) \neq 0$. Then there exists a value λ^* of λ that, together with x^*, satisfies

$$L_1(x^*, \lambda^*) = F'(x^*) - \lambda^* G'(x^*) = 0 \tag{1}$$

$$L_2(x^*, \lambda^*) = c - G(x^*) \geq 0 \tag{2}$$

$$\lambda^* \geq 0 \tag{3}$$

$$\lambda^*[c - G(x^*)] = 0 \tag{4}$$