Problem Set 9

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2023

Due Thursday, November 9

1. Natural Resource Depletion

Let c_t denote society's consumption of an exhaustible natural resource at each date $t = 0, 1, 2, ...$, and suppose that a representative consumer gets utility from this resource as described by

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t), \tag{1}$$

where the discount factor satisfies $0 < \beta < 1$. Let s_t denote the stock of the resource that remains at the beginning of each period $t = 0, 1, 2, ...$ Since the resource in nonrenewable, this stock evolves according to the constraint

$$s_t - c_t \geq s_{t+1}, \tag{2}$$

which indicates that consumption during period t just subtracts from the stock that is left during period $t + 1$.

a. A social planner chooses sequences $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ to maximize the representative consumer's utility in (1) subject to the constraint in (2), which must hold for all periods $t = 0, 1, 2, ...$, taking the initial stock s_0 as given. Probably, the easiest way to solve this problem is to use the method of Lagrange multipliers. Accordingly, write down the first-order conditions that describe the planner's optimal choices of c_t and s_t.

b. What do these first-order conditions tell you about the optimal path for consuming an exhaustible resource? Should c_t rise, fall, or stay the same over time?

c. It turns out that the transversality condition for this infinite-horizon problem implies that the optimal sequence $\{s_t\}_{t=0}^{\infty}$ must satisfy

$$\lim_{T \to \infty} s_{T+1} = 0.$$

See if you can use the first-order conditions that you derived in part (a), together with the binding constraint

$$s_{t+1} = s_t - c_t$$

and the transversality condition to solve for the optimal path $\{c_t\}_{t=0}^{\infty}$ for consumption of the exhaustible resource.

d. For the sake of completeness, see if you can also re-derive the optimality conditions that you obtained in part (a), using the Hamiltonian and the maximum principle instead.

2. Life Cycle Saving

Consider a consumer who is employed for $T+1$ periods: $t = 0, 1, ..., T$. During each period of employment, the consumer receives labor income w_t, which as the notation indicates can vary over time. Let k_t denote the consumer's stock of assets at the beginning of period t, and assume that $k_0 = 0$, so that the consumer begins his or her career with no assets. For all $t = 1, 2, ..., T$, k_t can be negative; that is, the consumer is allowed to borrow. However, the consumer must eventually save for retirement, a requirement that is captured by imposing the constraint

$$k_{T+1} \geq k^* > 0$$

on the terminal value of the stock of wealth.

Let r_t be the interest rate earned on savings, or paid on debt, during each period $t = 0, 1..., T$; again as the notation suggests, this interest rate can vary over time. Then the consumer's stock of assets evolves according to

$$k_{t+1} = k_t + w_t + r_t k_t - c_t$$

during each period $t = 0, 1, ..., T$. Allowing for the possibility of free disposal of wealth, which of course will never be optimal, these constraints can be written as

$$w_t + r_t k_t - c_t \geq k_{t+1} - k_t \tag{3}$$

for all $t = 0, 1, ..., T$.

So far, the set up of this problem generalizes the one that we studied in class by making labor income and the interest rate time-varying. Suppose, too, that the consumer's utility function also takes the more general, constant relative risk aversion form

$$\sum_{t=0}^{T} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right), \tag{4}$$

where $\sigma > 0$ and $0 < \beta < 1$.

The consumer's problem can now be stated as: choose sequences $\{c_t\}_{t=0}^{T}$ and $\{k_t\}_{t=1}^{T+1}$ to maximize the utility function (4) subject to the constraints $k_0 = 0$ given, (3) for all $t = 0, 1, 2, \ldots$, and $k_{T+1} \geq k^*$.

a. To solve the consumer's problem using the maximum principle, begin by setting up the maximized Hamiltonian.

b. Then write down the conditions that, according to the maximum principle, characterize the solution to the consumer's dynamic optimization problem.

c. Use your results from above to show how the optimal growth rate of consumption, c_t/c_{t-1} between periods $t-1$ and t depends on the preference parameters σ and β and on the interest rate r_t.