

Problem Set 9

ECON 772001 - Math for Economists
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Practice for the Midterm – Not Collected or Graded

1. Natural Resource Depletion

Let c_t denote society's consumption of an exhaustible natural resource at each date $t = 0, 1, 2, \dots$, and suppose that a representative consumer gets utility from this resource as described by

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t), \quad (1)$$

where the discount factor satisfies $0 < \beta < 1$. Let s_t denote the stock of the resource that remains at the beginning of each period $t = 0, 1, 2, \dots$. Since the resource is nonrenewable, this stock evolves according to the constraint

$$s_t - c_t \geq s_{t+1}, \quad (2)$$

which indicates that consumption during period t just subtracts from the stock that is left during period $t + 1$.

- Consider a social planner, who chooses sequences $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ to maximize the representative consumer's utility in (1) subject to the constraint in (2), which must hold for all periods $t = 0, 1, 2, \dots$, taking the initial stock s_0 as given. Probably, the easiest way to solve this problem is to use the method of Lagrange multipliers. Accordingly, write down the first-order conditions that describe the planner's optimal choices of c_t and s_t .
- What do these first-order conditions tell you about the optimal path for consuming an exhaustible resource? Should c_t rise, fall, or stay the same over time?
- It turns out that the transversality condition for this infinite-horizon problem implies that the optimal sequence $\{s_t\}_{t=0}^{\infty}$ must satisfy

$$\lim_{T \rightarrow \infty} s_{T+1} = 0.$$

See if you can use the first-order conditions that you derived in part (a), together with the binding constraint

$$s_{t+1} = s_t - c_t$$

and the transversality condition to solve the optimal path $\{c_t\}_{t=0}^{\infty}$ for consumption of the exhaustible resource.

- For the sake of completeness, see if you re-derive the same optimality conditions that you obtained in part (a), using the Hamiltonian and the maximum principle instead.

2. Life Cycle Saving

Consider a consumer who is employed for $T + 1$ periods: $t = 0, 1, \dots, T$. During each period of employment, the consumer receives labor income w_t , which as the notation indicates can vary over time. Let k_t denote the consumer's stock of assets at the beginning of period t , and assume that $k_0 = 0$, so that the consumer begins his or her career with no assets. For all $t = 1, 2, \dots, T$, k_t can be negative; that is, the consumer is allowed to borrow. However, the consumer must eventually save for retirement, a requirement that is captured by imposing the constraint

$$k_{T+1} \geq k^* > 0$$

on the terminal value of the stock of wealth.

Let r_t be the interest rate earned on savings, or paid on debt, during each period $t = 0, 1, \dots, T$; again as the notation suggests, this interest rate can vary over time. Then the consumer's stock of assets evolves according to

$$k_{t+1} = k_t + w_t + r_t k_t - c_t$$

during each period $t = 0, 1, \dots, T$. Allowing for the possibility of free disposal of wealth, which of course will never be optimal, these constraints can be written as

$$w_t + r_t k_t - c_t \geq k_{t+1} - k_t \quad (3)$$

for all $t = 0, 1, \dots, T$.

So far, the set up of this problem generalizes the one that we studied in class by making labor income and the interest rate time-varying. Suppose, too, that the consumer's utility function also takes the more general, constant relative risk aversion form

$$\sum_{t=0}^T \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right), \quad (4)$$

where $\sigma > 0$ and $0 < \beta < 1$.

The consumer's problem can now be stated as: choose sequences $\{c_t\}_{t=0}^T$ and $\{k_t\}_{t=1}^{T+1}$ to maximize the utility function (4) subject to the constraints $k_0 = 0$ given, (3) for all $t = 0, 1, 2, \dots$, and $k_{T+1} \geq k^*$.

- To solve the consumer's problem using the maximum principle, begin by setting up the maximized Hamiltonian.
- Then write down the conditions that, according to the maximum principle, characterize the solution to the consumer's dynamic optimization problem.
- Use your results from above to show how the optimal growth rate of consumption, c_t/c_{t-1} between periods $t - 1$ and t depends on the preference parameters σ and β and on the interest rate r_t .