

Problem Set 8

ECON 772001 - Math for Economists
Boston College, Department of Economics

Peter Ireland
Fall 2023

For Extra Practice - Not Collected or Graded

1. Optimal Growth

Note that the utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma},$$

with $\sigma > 0$, nests the logarithmic function

$$u(c) = \ln(c)$$

as the special case in which $\sigma = 1$. To see this, let

$$g(\sigma) = c^{1-\sigma}$$

for any given value of c . Then

$$\frac{g'(\sigma)}{g(\sigma)} = \frac{d}{d\sigma} \ln(g(\sigma)) = \frac{d}{d\sigma} (1-\sigma) \ln(c) = -\ln(c)$$

and therefore

$$\frac{d}{d\sigma} c^{1-\sigma} = g'(\sigma) = -\ln(c)g(\sigma) = -\ln(c)c^{1-\sigma}.$$

L'Hôpital's rule then leads to the result:

$$\lim_{\sigma \rightarrow 1} \frac{c^{1-\sigma} - 1}{1-\sigma} = \lim_{\sigma \rightarrow 1} \frac{-\ln(c)c^{1-\sigma}}{-1} = \ln(c) \lim_{\sigma \rightarrow 1} c^{1-\sigma} = \ln(c).$$

Moreover, since the coefficient of relative risk aversion implied by the more general utility function is

$$-\frac{cu''(c)}{u'(c)} = \sigma,$$

the log utility function now appears as a special case, in which the constant coefficient of relative risk aversion equals one.

With all this in mind, consider a version of the Ramsey model that is identical to the one we studied in class, but in which the representative consumer's utility function is generalized, as above, to allow for a constant coefficient of relative risk aversion that differs from one. As in class, let $k(t)$ denote the capital stock and $c(t)$ denote consumption at each period $t \in [0, \infty)$. Let output be produced using capital according to the production function $k(t)^\alpha$, with $0 < \alpha < 1$, and let $\delta > 0$ denote the depreciation rate for capital.

Then the representative consumer or social planner chooses functions $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ to maximize

$$\int_0^{\infty} e^{-\rho t} \left[\frac{c(t)^{1-\sigma} - 1}{1-\sigma} \right] dt$$

subject to

$$k(0) \text{ given}$$

and

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$, where $\rho > 0$ is the discount rate and σ is the constant coefficient of relative risk aversion.

- a. To solve this problem using the maximum principle, begin by writing down the expression for the maximized current value Hamiltonian.
- b. Next, write down the first order condition and the pair of differential equations that, according to maximum principle, characterize the solution to the dynamic optimization problem.
- c. As in class, combine these optimality conditions so as to obtain two differential equations in the two unknown functions $c(t)$ and $k(t)$ that solve the dynamic optimization problem – differential equations that make no reference to objects like Lagrange multipliers that lack a direct economic interpretation.
- d. Finally, as in class, draw a phase diagram to illustrate the key properties of the unique solutions for $c(t)$ and $k(t)$ that satisfy the initial condition $k(0)$ given and the terminal, or transversality condition, which in this more general version of the model is

$$\lim_{T \rightarrow \infty} e^{-\rho T} c(T)^{-\sigma} k(T) = 0.$$