Problem Set 8

ECON 772001 - Math for Economists | Peter Ireland
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For Extra Practice - Not Collected or Graded

1. Optimal Growth

Note that the utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma},$$

with $\sigma > 0$, nests the logarithmic function

$$u(c) = \ln(c)$$

as the special case in which $\sigma = 1$. To see this, let

$$g(\sigma) = c^{1-\sigma}$$

for any given value of c. Then

$$\frac{g'(\sigma)}{g(\sigma)} = \frac{d}{d\sigma}\ln(g(\sigma)) = \frac{d}{d\sigma}(1-\sigma)\ln(c) = -\ln(c)$$

and therefore

$$\frac{d}{d\sigma}c^{1-\sigma} = g'(\sigma) = -\ln(c)g(\sigma) = -\ln(c)c^{1-\sigma}.$$

L'Hôpital's rule then leads to the result:

$$\lim_{\sigma\to 1}\frac{c^{1-\sigma}-1}{1-\sigma} = \lim_{\sigma\to 1}\frac{-\ln(c)c^{1-\sigma}}{-1} = \ln(c)\lim_{\sigma\to 1}c^{1-\sigma} = \ln(c).$$

Moreover, since the coefficient of relative risk aversion implied by the more general utility function is

$$-\frac{cu''(c)}{u'(c)} = \sigma,$$

the log utility function now appears as a special case, in which the constant coefficient of relative risk aversion equals one.

With all this in mind, consider a version of the Ramsey model that is identical to the one we studied in class, but in which the representative consumer's utility function is generalized, as above, to allow for a constant coefficient of relative risk aversion that differs from one. As in class, let $k(t)$ denote the capital stock and $c(t)$ denote consumption at each period $t \in [0,\infty)$. Let output be produced using capital according to the production function $k(t)^\alpha$, with $0 < \alpha < 1$, and let $\delta > 0$ denote the depreciation rate for capital.

Then the representative consumer or social planner chooses functions $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ to maximize

$$\int_0^\infty e^{-\rho t} \left[\frac{c(t)^{1-\sigma} - 1}{1 - \sigma} \right] \mathrm{d}t$$

subject to

$$k(0) \text{ given}$$

and

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$, where $\rho > 0$ is the discount rate and σ is the constant coefficient of relative risk aversion.

a. To solve this problem using the maximum principle, begin by writing down the expression for the maximized current value Hamiltonian.

b. Next, write down the first order condition and the pair of differential equations that, according to maximum principle, characterize the solution to the dynamic optimization problem.

c. As in class, combine these optimality conditions so as to obtain two differential equations in the two unknown functions $c(t)$ and $k(t)$ that solve the dynamic optimization problem – differential equations that make no reference to objects like Lagrange multipliers that lack a direct economic interpretation.

d. Finally, as in class, draw a phase diagram to illustrate the key properties of the unique solutions for $c(t)$ and $k(t)$ that satisfy the initial condition $k(0)$ given and the terminal, or transversality condition, which in this more general version of the model is

$$\lim_{T \to \infty} e^{-\rho T} c(T)^{-\sigma} k(T) = 0.$$