

## Problem Set 7

ECON 772001 - Math for Economists  
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For Extra Practice – Not Collected or Graded

### 1. The Permanent Income Hypothesis

The permanent income hypothesis describes how a forward-looking consumer optimally saves or borrows to smooth out his or her consumption in the face of a fluctuating income stream. This problem formalizes the permanent income hypothesis using a two-period model. So consider a consumer who lives for two periods, earning income  $w_0$  during period  $t = 0$  and  $w_1$  during period  $t = 1$ . Let  $c_0$  and  $c_1$  denote his or her consumption during periods  $t = 0$  and  $t = 1$  and let  $s$  denote his or her amount saved (or borrowed, if negative) during period  $t = 0$ . Suppose that savings earn interest between  $t = 0$  and  $t = 1$  at the constant rate  $r$ . Then the consumer faces the budget constraints

$$w_0 \geq c_0 + s \tag{1}$$

at  $t = 0$  and

$$w_1 + (1 + r)s \geq c_1 \tag{2}$$

at  $t = 1$ . Finally, suppose that the consumer's preferences are described by the utility function

$$\ln(c_0) + \beta \ln(c_1), \tag{3}$$

where the discount factor lies between zero and one:  $0 < \beta < 1$ .

- a. Find the values for  $c_0^*$ ,  $c_1^*$ , and  $s^*$  that solve the consumer's problem – choose  $c_0$ ,  $c_1$ , and  $s$  to maximize the utility function in (3) subject to the constraints in (1) and (2) – in terms of the parameters  $w_0$ ,  $w_1$ ,  $r$ , and  $\beta$ .
- b. Suppose now that the market interest rate and the consumer's discount factor satisfy  $\beta(1 + r) = 1$ . Although it might seem that this condition will hold only through a rare coincidence, in fact as problem set 6 showed, it is often satisfied in more elaborate general equilibrium models that feature a large numbers of consumers who borrow and lend in a competitive market. With this extra condition imposed, what do your solutions from part (a) above imply for the optimal behavior of consumption: does it rise, fall, or stay the same moving from period  $t = 0$  to period  $t = 1$ ?
- c. Continuing to assume that  $\beta(1 + r) = 1$ , what determines whether the consumer will borrow (choosing  $s < 0$ ) or save (choosing  $s > 0$ ) during period  $t = 0$ ?

## 2. Habit Formation

Consider the behavior of a consumer that faces the same budget constraints (1) and (2) shown above, but instead of (3) has the utility function

$$\ln(c_0) + \beta \ln(c_1 - \gamma c_0), \quad (4)$$

where  $0 < \gamma < 1$ . The preferences represented by this utility function display “habit formation” or capture an “addictive” element of consumption by implying that the more the individual consumes during period  $t = 0$  the more he or she will want to consume during period  $t = 1$ .

- a. Assuming once more that  $\beta$  and  $r$  are separate parameters, find the values for  $c_0^*$ ,  $c_1^*$ , and  $s^*$  that solve the consumer’s problem – choose  $c_0$ ,  $c_1$ , and  $s$  to maximize the utility function in (4) subject to the constraints in (1) and (2) – in terms of the parameters  $w_0$ ,  $w_1$ ,  $r$ ,  $\beta$ , and  $\gamma$ .
- b. Now assume that  $\beta(1 + r) = 1$  holds again. What happens to the optimal behavior of consumption now: is  $c_1^*$  larger than, smaller than, or the same as  $c_0^*$ ?

## 3. Durable Consumption

Now replace the utility function in (4) with

$$\ln(c_0) + \beta \ln(c_1 + \theta c_0), \quad (5)$$

where  $0 < \theta < 1$ . This specification might capture an aspect of “durability” in the consumption good, whereby a purchase made at  $t = 0$  continues to yield utility at  $t = 1$ , in which case the new parameter  $\theta$  can be interpreted as measuring the depreciation rate for the durable item.

- a. Assuming again that  $\beta$  and  $r$  are separate parameters, find the values for  $c_0^*$ ,  $c_1^*$ , and  $s^*$  that solve the consumer’s problem – choose  $c_0$ ,  $c_1$ , and  $s$  to maximize the utility function in (5) subject to the constraints in (1) and (2) – in terms of the parameters  $w_0$ ,  $w_1$ ,  $r$ ,  $\beta$ , and  $\theta$ .
- b. Now assume for the last time that  $\beta(1 + r) = 1$  holds. What happens to the optimal behavior of consumption in this case: is  $c_1^*$  larger than, smaller than, or the same as  $c_0^*$ ?