Problem Set 6

ECON 772001 - Math for Economists
Boston College, Department of Economics

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Fall 2020

For Extra Practice – Not Collected or Graded

This problem set asks you to solve a series of dynamic problems that characterize the optimal lending and borrowing behavior of individual consumers, the determination of the equilibrium interest rate, and the link between equilibrium and optimal resource allocations in an economy in which, for simplicity, it is assumed that all agents live for only two periods. Each consumer is "young" during period $t = 0$ and "old" during period $t = 1$. Consumers are of two types, called "lenders" and "borrowers" for reasons that will (hopefully) become clear when their individual optimization problems are described next.

1. Optimal Lending

Consider first the behavior of a "lender" who receives an endowment consisting of one unit of the economy's single consumption good during period $t = 0$ when he or she is young. Let c_0^L denote the consumption of this agent during period $t = 0$ when young and let s^L denote the saving of this agent during period $t = 0$ when young. Assume that this lender earns interest on his or her savings at the rate r between periods $t = 0$ and $t = 1$. This lender receives no endowment during period $t = 1$, and hence must finance consumption c_1^L when old exclusively from his or her savings. Assume that this consumer has an additively time-separable utility function with single-period utility that is logarithmic in form and that utility at when old is discounted relative to utility when young using the discount factor β, which satisfies $0 < \beta < 1$. Now the lender's dynamic optimization problem can be stated formally as:

$$\max_{c_0^L, c_1^L, s^L} \ln(c_0^L) + \beta \ln(c_1^L) \text{ subject to } 1 \geq c_0^L + s^L \text{ and } (1 + r)s^L \geq c_1^L.$$

Note that nonnegativity conditions are not imposed on any of the choice variables, since the form of the utility function implies that the consumer will always want to consume at least some positive amount during each period and since negative values for s^L can be interpreted meaningfully as describing situations where the consumer is borrowing instead of saving.

a. Define the Lagrangian for this consumer's problem, letting λ_0^L denote the multiplier on the budget constraint for period $t = 0$ and λ_1^L denote the multiplier on the budget constraint for period $t = 1$.

b. Next, write down the conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the lender's optimal choices of c_0^{L*}, c_1^{L*}, and s^{L*} together with the associated values of λ_0^{L*} and λ_1^{L*}.

c. Use the first-order conditions that you derived above, together with the two constraints, which will always bind at the optimum, to solve for c_0^{L*}, c_1^{L*}, and s^{L*} in terms of the parameters β and r.

2. Optimal Borrowing

Consider next the behavior of a "borrower" who receives no endowment during period $t = 0$ when young but, instead, receives one unit of the consumption good as an endowment during period $t = 1$ when old. Extending the notation from above, this borrower's dynamic optimization problem can be stated as

$$\max_{c_0^B, c_1^B, s^B} \ln(c_0^B) + \beta \ln(c_1^B) \text{ subject to } 0 \geq c_0^B + s^B \text{ and } 1 + (1+r)s^B \geq c_1^B.$$

Note again that the borrower's choice of s^B is allowed to be negative, so that he or she can borrow to finance his or her consumption c_0^B during period $t = 0$ when young, provided that he or she repays this loan with interest during period $t = 1$ when old.

a. Define the Lagrangian for this consumer's problem, letting λ_0^B denote the multiplier on the budget constraint for period $t = 0$ and λ_1^B denote the multiplier on the budget constraint for period $t = 1$.

b. Next, write down the conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the borrower's optimal choices of c_0^{B*}, c_1^{B*}, and s^{B*} together with the associated values of λ_0^{B*} and λ_1^{B*}.

c. Use the first-order conditions that you derived above, together with the two constraints, which will always bind at the optimum, to solve for c_0^{B*}, c_1^{B*}, and s^{B*} in terms of the parameters β and r.

3. Equilibrium Allocations

Now suppose that economy consists of an equal number n of lenders and borrowers. Equilibrium in the market for lending and borrowing then requires that

$$ns^{L*} + ns^{B*} = 0.$$

Keeping in mind that borrowing corresponds to "negative saving" and exploiting the simplifying assumption that there are equal numbers of each type of agent, this market-clearing condition requires that the interest rate r adjust so that each individual lender optimally chooses to lend exactly the same amount that each individual borrower chooses to borrow.

a. Substitute your solutions for s^{L*} and s^{B*} into this market-clearing condition to determine how the equilibrium interest rate r depends on the remaining parameter β. Is this equilibrium interest rate higher or lower when individual consumers are more or less patient?

b. Next, substitute your solution for the equilibrium interest rate back into your solutions for c_0^{L*}, c_1^{L*}, c_0^{B*}, c_1^{B*} to find the equilibrium values for each of these variables.

c. In equilibrium, does the lender's consumption rise, fall, or stay the same moving from period $t = 0$ to period $t = 1$? Does the borrower's consumption rise, fall, or stay the same?

d. Who gets to consume more during period $t = 0$: lenders or borrowers? Who gets to consume more during period $t = 1$. Can you explain why?

4. Optimal Allocations

Now suppose that this same economy is run instead by a benevolent social planner, who has the power to take some of the lenders' endowment away and give it to the borrowers during period $t = 0$ and to similarly take some of the borrowers' endowment away and give it to the lenders during period $t = 1$ if such resource transfers turn out to make everyone better off. Assuming that the social planner treats all agents of a given type the same during each period, let c_0^L and c_0^B denote the amount of consumption that planner gives to each lender and borrower during period $t = 0$ and, similarly, let c_1^L and c_1^B denote the amount of consumption that planner gives to each lender and borrower during period $t = 1$. The fact that there are n agents of each type and the fact that only lenders get an endowment when young and only borrowers get an endowment when old means that the planner faces the aggregate resource constraints

$$n \geq nc_0^L + nc_0^B$$

during period $t = 0$ and

$$n \geq nc_1^L + nc_1^B$$

during period $t = 1$. Finally, suppose that the planner chooses c_0^L, c_0^B, c_1^L , and c_1^B to maximize a weighted average of utilities enjoyed by a representative lender and a representative borrower:

$$\omega[\ln(c_0^L) + \beta \ln(c_1^L)] + (1 - \omega)[\ln(c_0^B) + \beta \ln(c_1^B)]$$

subject to the two resource constraints, where the weight ω assigned to the lender's utility satisfies $0 < \omega < 1$.

a. Define the Lagrangian for the planner's problem, letting λ_0^P denote the multiplier on the resource constraint for period $t = 0$ and λ_1^P denote the multiplier on the resource constraint for period $t = 1$. In doing this, it might help to start by dividing both sides of each constraint through by the population size n, since doing so doesn't change the economic meaning of the constraints but will simplify the math.

b. Next, write down the conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the planners's optimal choices of c_0^{L*}, c_0^{B*}, c_1^{L*}, and c_1^{B*} together with the associated values of λ_0^{P*} and λ_1^{P*}.

c. Use the first-order conditions that you derived above, together with the two resource constraints, which will always bind at the optimum, to solve for c_0^{L*}, c_0^{B*}, c_1^{L*}, and c_1^{B*} in terms of the parameters β and ω.

d. Do these optimal allocations call for the lender's consumption to rise, fall, or stay the same moving from period $t = 0$ to period $t = 1$? Does the borrower's consumption rise, fall, or stay the same from $t = 0$ to $t = 1$?

e. What value for ω makes the optimal allocations you just characterized above coincide with the equilibrium allocations you solved for previously?