Problem Set 4

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2020

For Extra Practice – Not Collected or Graded

1. Upper Hemicontinuity

Let $\Theta = [0,1] \subseteq \mathbf{R}$ and $X = [0,1] \subseteq \mathbf{R}$, and consider the correspondence $G : \Theta \to X$ defined by

$$G(\theta) = \begin{cases} (0,1] & \text{for} \quad \theta = 0 \\ (0,\theta] & \text{for} \quad \theta \in (0,1]. \end{cases}$$

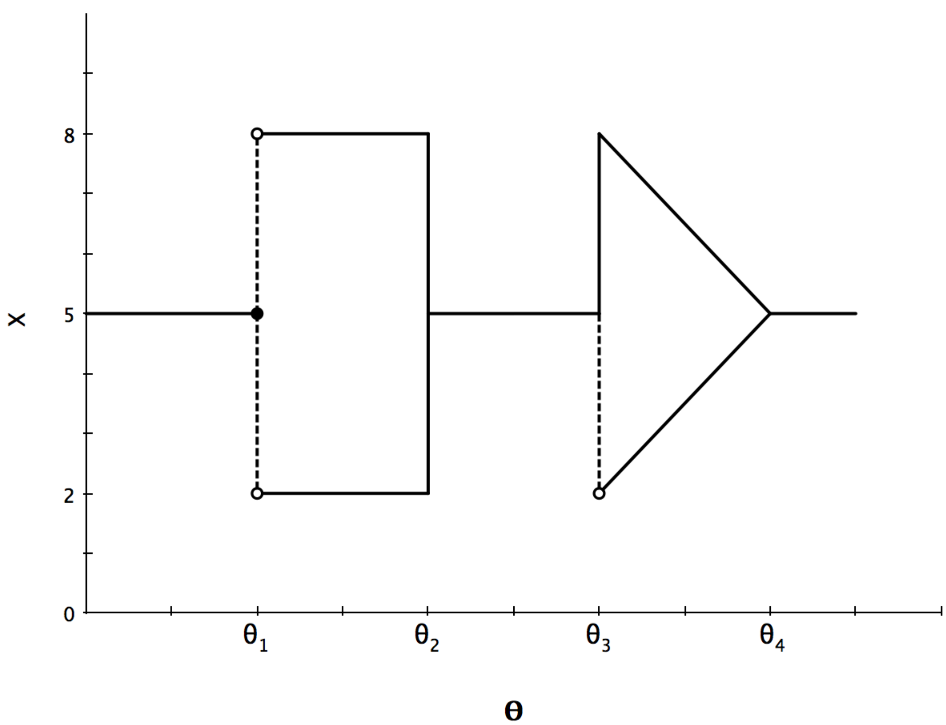
a. Draw a graph of $G(\theta)$ with Θ on the horizontal axis and X on the vertical axis.

b. Is G upper hemicontinuous?

c. Let $\{\theta_j\}$ be a sequence with $\theta_j \to 0$ and let $\{x_j\}$ be a sequence with $x_j \in G(\theta_j)$ for all j. Does there exist a convergent subsequence $\{x_{j_k}\}$ of $\{x_j\}$ such that $x_{j_k} \to x \in G(0)$?

2. Upper and Lower Hemicontinuity

The graph below shows the graph of a compact-valued correspondence $G : \Theta \to X$, where $\Theta \subseteq \mathbf{R}$ and $X \subseteq \mathbf{R}$.



The graph reveals, in particular, that

$$G(\theta) = \begin{cases} 5 & \text{for} \quad \theta_1 \geq \theta \\ [2,8] & \text{for} \quad \theta_2 \geq \theta > \theta_1 \\ 5 & \text{for} \quad \theta_3 > \theta > \theta_2 \\ [5,8] & \text{for} \quad \theta = \theta_3 \\ \left[2 + \frac{3(\theta-\theta_3)}{\theta_4-\theta_3}, 8 - \frac{3(\theta-\theta_3)}{\theta_4-\theta_3}\right] & \text{for} \quad \theta_4 > \theta > \theta_3 \\ 5 & \text{for} \quad \theta \geq \theta_4 \end{cases}$$

a. Is G upper hemicontinuous at θ_1? Is G lower hemicontinuous at θ_1?

b. Is G upper hemicontinuous at θ_2? Is G lower hemicontinuous at θ_2?

c. Is G upper hemicontinuous at θ_3? Is G lower hemicontinuous at θ_3?

d. Is G upper hemicontinuous at θ_4? Is G lower hemicontinuous at θ_4?

3. Pareto Optimal Allocations

Consider an economy at a point in time when the aggregate stock of physical capital is fixed at $k \in K = [\underline{k}, \bar{k}]$, where $0 < \underline{k} < \bar{k} < \infty$. These k units of capital can be used to produce $f(k)$ units of output, where the aggregate production function $f : K \to (0, \infty)$ is continuous.

Suppose this economy is populated by two consumers. Consumer 1 enjoys utility $u(c_1)$ from consuming c_1 units of output and consumer 2 enjoys utility $v(c_2)$ from consuming c_2 units of output, where the utility functions u and v are both continuous.

A social planner in this economy finds Pareto optimal allocations by choosing c_1 and c_2 from the constraint set

$$G(k) = \left\{(c_1, c_2) \in \mathbf{R}^2 \mid f(k) \geq c_1 + c_2,\ c_1 \geq 0,\ c_2 \geq 0\right\}$$

to maximize the weighted sum of the consumers' utilities

$$\omega u(c_1) + (1 - \omega)v(c_2),$$

where $0 < \omega < 1$.

As a weighted sum of continuous utility functions, the planner's objective function is also continuous. See if you can adapt the arguments we applied in class to a consumer's utility maximization to show that, for this planner's problem, the correspondence $G : K \to \mathbf{R}^2$ is compact valued and continuous.

Berge's maximum theorem then implies that the maximum value function

$$W(k) = \max_{(c_1,c_2) \in G(k)} \omega u(c_1) + (1 - \omega)v(c_2)$$

is well-defined and continuous and that the optimal policy correspondence

$$c^*(k) = \{(c_1^*, c_2^*) \in G(k) \mid \omega u(c_1^*) + (1 - \omega)v(c_2^*) = W(k)\}$$

is nonempty, compact-valued, and upper hemicontinuous.