Problem Set 3

ECON 772001 - Math for Economists | Peter Ireland
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Many famous results from microeconomic theory are now understood to be special cases of the envelope theorem. This problem set will ask you to invoke the envelope theorem repeatedly to "prove" some of these results.

1. Hotelling's Lemma

Consider a firm that produces output y with capital k and labor l according to the technology described by

$$f(k, l) \geq y.$$

The firm sells each unit of output at the price p, rents each unit of capital at the rate r, and hires each unit of labor at the wage w. Hence it chooses y, k, and l to maximize profits

$$py - rk - wl$$

subject to the technological constraint just shown above.

a. Set up the Lagrangian for this problem, letting λ denote the multiplier on the constraint.

b. Next, write down the conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values y^*, k^*, and l^* that solve the firm's problem, together with the associated value λ^* for the multiplier.

c. Assume that the output and input prices p, r, and w and the production function f are such that it is possible to solve uniquely for the values of y^*, k^*, l^*, and λ^* in terms of the parameters p, r, and w. Then the function $y^*(p, r, w)$ describing the optimal level of output represents the firm's supply function, and functions $k^*(p, r, w)$ and $l^*(p, r, w)$ describing the optimal inputs are the firm's factor demand curves. Along with these functions, define the firm's profit function as

$$\pi(p, r, w) = \max_{y,k,l} py - rk - wl \text{ subject to } f(k, l) \geq y.$$

Hotelling's lemma says that the derivative of this profit function with respect to the output price coincides with the supply function and that the derivatives of the profit function with respect to each input price, multiplied by -1, coincide with the associated factor demand curves. Use the definitions of all these functions to show that Hotelling's lemma is really just a special case of the envelope theorem.

2. Shephard's Lemma

Closely related to the profit maximization problem from above is the corresponding cost minimization problem in which the same firm chooses capital k and labor l inputs to minimize the cost of producing y units of output:

$$\min_{k,l} rk + wl \text{ subject to } f(k,l) \geq y.$$

Note that in this version of the firm's problem, y is a parameter instead of a choice variable.

a. Set up the Lagrangian for this problem, letting μ denote the multiplier on the constraint.

b. Next, write down the conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values k^* and l^* that solve the firm's problem, together with the associated value μ^* for the multiplier.

c. Assume that the output requirement y, the input prices r and w, and the production function f are such that it is possible to solve uniquely for the values of k^*, l^*, and μ^* in terms of the parameters r, w, and y. Then the functions $k^*(r,w,y)$ and $l^*(r,w,y)$ describing the optimal inputs can be called the firm's conditional factor demand curves. Along with these functions, define the firm's cost function as

$$c(r,w,y) = \min_{k,l} rk + wl \text{ subject to } f(k,l) \geq y.$$

Shephard's lemma says that the derivatives of this cost function with respect to each input price coincides with the associated conditional factor demand curve. Use the definitions of all these functions to show that Shephard's lemma is really just a special case of the envelope theorem.

3. Roy's Identity

Now consider a utility-maximizing consumer who solves

$$\max_{c_1,c_2} U(c_1,c_2) \text{ subject to } I \geq p_1 c_1 + p_2 c_2,$$

where c_1 and c_2 denote consumption of two goods, U is the utility function, I is income, and p_1 and p_2 are the prices of the two goods.

a. Set up the Lagrangian for this problem, letting λ denote the multiplier on the constraint.

b. Next, write down the conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values c_1^* and c_2^* that solve the consumer's problem, together with the associated value λ^* for the multiplier.

c. Assume that income I, and goods prices p_1 and p_2, and the utility function U are such that it is possible to solve uniquely for the values of c_1^*, c_2^*, and λ^* in terms of the parameters I, p_1, and p_2. Recall that the function $c_1^*(p_1, p_2, I)$ and $c_1^*(p_1, p_2, I)$ describing these optimal choices are the Marshallian demand curves for the two goods. Along with these functions, define the indirect utility function

$$v(p_1, p_2, I) = \max_{c_1, c_2} U(c_1, c_2) \text{ subject to } I \geq p_1 c_1 + p_2 c_2.$$

Roy's identity says that the Marshallian demand curves are related to the derivatives of the indirect utility function via

$$c_i^*(p_1, p_2, I) = \frac{-v_i(p_1, p_2, I)}{v_3(p_1, p_2, I)}$$

for each $i = 1, 2$. Use the definitions of all these functions to show that Roy's identity follows directly from the envelope theorem.

4. The Slutsky Equation

Closely related to the utility maximization problem from above is the corresponding expenditure minimization problem in which the same consumer chooses consumptions c_1 and c_2 to minimize the cost of achieving a given utility level $\bar{U}$:

$$\min_{c_1, c_2} p_1 c_1 + p_2 c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U}.$$

Note that in this version of the problem, $\bar{U}$ replaces I as one of the parameters.

a. Set up the Lagrangian for this problem, letting μ denote the multiplier on the constraint.

b. Next, write down the conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values c_1^* and c_2^* that solve this problem, together with the associated value μ^* for the multiplier.

c. Assume that the utility level $\bar{U}$, goods prices p_1 and p_2, and the utility function U are such that it is possible to solve uniquely for the values of c_1^*, c_2^*, and μ^* in terms of the parameters $\bar{U}$, p_1, and p_2. Now the functions $c_1^* = h_1^*(p_1, p_2, \bar{U})$ and $c_2^* = h_2^*(p_1, p_2, \bar{U})$ describing the optimal choices are the Hicksian demand curves for the two goods (note that h's instead of c's are being used here to help distinguish these Hicksian demand curves from the Marshallian demand curves defined by the solution to the utility maximization problem). Along with these functions, define the expenditure function

$$e(p_1, p_2, \bar{U}) = \min_{c_1, c_2} p_1 c_1 + p_2 c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U}.$$

Under most circumstances, it will be the case that the Marshallian and Hicksian demand curves coincide at the point where $I = e(p_1, p_2, \bar{U})$, so that the income I from the utility maximization problem equals the expenditure required to attain the utility level $v(p_1, p_2, I) = \bar{U}$. This result can be summarized by stating that

$$h_i^*(p_1, p_2, \bar{U}) = c_i^*(p_1, p_2, e(p_1, p_2, \bar{U}))$$

for all values of p_1, p_2, and $\bar{U}$ and each $i = 1, 2$. Use this last relationship, together with the definitions of all of the functions and the envelope theorem, to show that under these same circumstances, the Marshallian and Hicksian demand curves must also satisfy the Slutsky equation

$$\frac{\partial c_i^*(p_1, p_2, I)}{\partial p_i} = \frac{\partial h_i^*(p_1, p_2, v(p_1, p_2, I))}{\partial p_i} - \frac{\partial c_i^*(p_1, p_2, I)}{\partial I}\, c_i^*(p_1, p_2, I)$$

for each $i = 1, 2$.