Problem Set 14

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2020

Practice for the Final – Not Collected or Graded

1. Stochastic Linear-Quadratic Dynamic Programming

This problem asks you to use dynamic programming to characterize the solution to a stochastic version of the linear-quadratic problem that you studied previously, under conditions of perfect foresight, in problem set 12. The problem is to choose contingency plans for a flow variable z_t for all $t = 0, 1, 2, ...$ and a stock variable y_t for all $t = 1, 2, 3, ...$ to maximize the objective function

$$E_0 \sum_{t=0}^{\infty} \beta^t (Ry_t^2 + Qz_t^2),$$

subject to the constraints y_0 given and

$$Ay_t + Bz_t + \varepsilon_{t+1} \geq y_{t+1}, \tag{1}$$

where $0 < \beta < 1$, $R < 0$, $Q < 0$, A, and B are all constant, known parameters and ε_{t+1} is an independently and identically distributed random shock that satisfies $E_t(\varepsilon_{t+1}) = 0$ and $E_t(\varepsilon_{t+1}^2) = \sigma^2$, that is, which has zero mean and variance σ^2. The flow variable z_t must be chosen at time t before ε_{t+1} is known, hence the constraint (1) must hold for all $t = 0, 1, 2, ...$ and for all possible realizations of ε_{t+1}.

a. Guess that the value function for this problem depends only on y_t and not ε_t and takes the specific form

$$v(y_t, \varepsilon_t) = v(y_t) = Py_t^2 + d,$$

where P and d are unknown constants. Using this guess, write down the Bellman equation for the problem. In doing this, it might be helpful to note that the assumptions that ε_{t+1} is iid and that y_t and z_t are known at time t imply that

$$\begin{aligned} E_t[(Ay_t + Bz_t + \varepsilon_{t+1})^2] &= E_t[(Ay_t + Bz_t)^2] + 2E_t[(Ay_t + Bz_t)\varepsilon_{t+1}] + E_t(\varepsilon_{t+1}^2) \\ &= (Ay_t + Bz_t)^2 + 2(Ay_t + Bz_t)E_t(\varepsilon_{t+1}) + E_t(\varepsilon_{t+1}^2) \\ &= (Ay_t + Bz_t)^2 + \sigma^2. \end{aligned}$$

b. Next, write down the first-order condition for z_t and the envelope condition for y_t.

c. Use your results from above to show that the unknown P must satisfy the same Riccati equation

$$P = R + \frac{\beta A^2 QP}{Q + \beta B^2 P}$$

that helps characterize the problem's solution in the nonstochastic case.

d. Finally, use your results from above to show how the new unknown parameter d that enters the value function for the stochastic problem depends on the value of P that solves the Riccati equation along with the model's other parameters.

2. Stochastic Growth

This problem asks you to use dynamic programming to solve a stochastic version of the optimal growth model in which physical capital depreciates fully between periods; as in the nonstochastic case, that assumption of full depreciation allows an explicit solution for the value function to be found via the guess-and-verify method. In this version of the model, the representative consumer chooses contingency plans for consumption c_t for all $t = 0, 1, 2, ...$ and physical capital k_t for all $t = 1, 2, 3, ...$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

with $0 < \beta < 1$, subject to the constraints k_0 given and

$$z_t k_t^{\alpha} \geq c_t + k_{t+1}, \tag{2}$$

where $0 < \alpha < 1$. In (2), z_t represents a shock to the productivity of capital. The value of z_t is known when c_t and k_{t+1} are chosen during period t, but the value of z_{t+1} is random and satisfies $E_t[\ln(z_{t+1})] = 0$ for all $t = 0, 1, 2, ...$ Then (2) must hold for all $t = 0, 1, 2, ...$ and all possible realizations of z_t.

a. Guess that the value function for this problem takes the form

$$v(k_t, z_t) = E + F \ln(k_t) + G \ln(z_t),$$

where E, F, and G are unknown constants. Using this guess, write down the Bellman equation for the problem. In doing this, it might be helpful to note that

$$E_t[\ln(z_t k_t^{\alpha} - c_t)] = \ln(z_t k_t^{\alpha} - c_t)$$

since all of the objects in this expression are known at time t and to recall that

$$E_t[\ln(z_{t+1})] = 0$$

by assumption.

b. Next, write down the first-order condition for c_t and the envelope condition for k_t.

c. Use your results from above to derive an equation that shows how the optimal choice of c_t depends on k_t, z_t, and the parameters α and β. Then substitute this expression into the binding constraint (2) to derive an equation that shows how the optimal choice of k_{t+1} depends on k_t, z_t, and the parameters α and β.

d. For the sake of completeness, write down solutions that show how the unknown constants E, F, and G depend on the parameters α and β.

3. Saving with a Random Return

This problem asks you to use dynamic programming to solve a representative consumer's problem when savings earn a random rate of return. Let A_t denote the consumer's assets at the beginning of each period $t = 0, 1, 2, ...$ During each period t, the consumer divides these assets up into an amount c_t to be consumed and an amount s_t to be saved. The consumer's savings earn interest at the gross rate R_{t+1}, where R_{t+1} is random, possibly serially correlated, and does not become known until the beginning of period $t + 1$. Thus, the consumer must choose s_t before knowing the realized value of R_{t+1}. The consumer takes his or her initial assets A_0 as given and chooses contingency plans for s_t for all $t = 0, 1, 2, ...$ and A_t for $t = 1, 2, 3, ...$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) = E_0 \sum_{t=0}^{\infty} \beta^t u(A_t - s_t),$$

with $0 < \beta < 1$, subject to the constraints

$$R_{t+1}s_t \geq A_{t+1},$$

which must hold for all $t = 0, 1, 2, ...$ and all possible realizations of R_{t+1}.

a. Write down the Bellman equation for this problem, using A_t as the state variable, s_t and the control variable, and allowing the value function for time t to depend on R_t as well as A_t.

b. Now suppose that the consumer's single-period utility function takes the form

$$u(c_t) = u(A_t - s_t) = \frac{(A_t - s_t)^{1-\sigma}}{1 - \sigma},$$

where $\sigma > 0$ and $\sigma \neq 0$. Suppose also that the random interest rate R_{t+1} is independently and identically distributed with

$$E_t(R_{t+1}^{1-\sigma}) = 1$$

for all $t = 0, 1, 2, ...$ Guess that under these conditions, the value function depends only on A_t and takes the specific form

$$v(A_t) = \frac{KA_t^{1-\sigma}}{1 - \sigma},$$

where K is an unknown constant. Use this guess, together with the assumptions about the form of $u(c_t)$ and the properties of R_{t+1}, to rewrite your Bellman equation from part (a) above. Then derive the first-order and envelope conditions for the consumer's problem.

c. Use your results from part (b) to solve for the unknown K in terms of the parameters β and σ.

d. Finally use your results from above to derive expressions that show how the optimal choices for c_t and s_t depend on the stock of assets A_t and the parameters β and σ.