Problem Set 13

ECON 772001 - Math for Economists   Peter Ireland

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Human Capital Accumulation and Economic Growth

This problem set asks you to use dynamic programming to characterize the solution to a variant of the model of human capital accumulation and economic growth studied by Robert Lucas, "On the Mechanics of Economic Development," *Journal of Monetary Economics*, July 1988, pp.3-42 and, long before that, by Hirofumi Uzawa, "Optimum Technical Change in an Aggregative Model of Economic Growth," *International Economic Review*, January 1965, pp.18-31. The specific version of that model considered here comes from Dirk Bethmann, "A Closed-Form Solution of the Uzawa-Lucas Model of Endogenous Growth," *Journal of Economics*, January 2007, pp.87-107.

In this model, a representative consumer divides up his or her time into an amount u_t devoted to education, technical training, and other activities that add to the stock of human capital and an amount $1 - u_t$ devoted to producing goods and services for consumption and investment in physical capital. Let h_t and k_t denote the stocks of human capital and physical capital at the beginning of each period $t = 0, 1, 2, ...$, let c_t denote the amount of output consumed during each period $t = 0, 1, 2, ...$, and assume that the two stocks evolve according to

$$\gamma u_t h_t \geq h_{t+1} \tag{1}$$

and

$$k_t^\alpha [(1 - u_t)h_t]^{1-\alpha} \geq c_t + k_{t+1} \tag{2}$$

for all $t = 0, 1, 2, ...$. Equation (1), with $\gamma > 1$, describes how human capital gets built up through time u_t allocated to schooling and training. Equation (2), with $0 < \alpha < 1$, describes how output of goods and services gets produced with physical capital, human capital, and time $1 - u_t$ allocated to production. Note that an assumption that physical capital depreciates fully in use during each period is also embedded into (2); as in our previous analysis of the neoclassical growth model with physical capital accumulation alone, this assumption is needed to be able to solve explicitly for the value function using the guess-and-verify method.

What makes this model more complicated than the ones that we've considered before is that there are two sequences for flow variables, $\{c_t\}_{t=0}^\infty$ and $\{u_t\}_{t=0}^\infty$, and two sequences for stock variables, $\{h_t\}_{t=1}^\infty$ and $\{k_t\}_{t=1}^\infty$, that must be chosen to maximize the utility function

$$\sum_{t=0}^\infty \beta^t \ln(c_t),$$

with $0 < \beta < 1$, subject to two sets of constraints, (1) and (2) for all $t = 0, 1, 2, ...$, taking two initial conditions h_0 and k_0 as given. Still, as the following questions will help you

confirm, our previous analysis generalizes in a natural way to accommodate these extra model features.

1. With two stock variables, the value function that enters into the Bellman equation for the representative consumer or social planner's problem must be defined as a function of both k_t and h_t. Similarly, the static optimization problem on the right-hand-side of the Bellman equation will involve the choice of two control variables: c_t and u_t. More specifically, the Bellman equation for the problem can be written as

$$v(k_t, h_t; t) = \max_{c_t, u_t} \ln(c_t) + \beta v(k_t^\alpha[(1 - u_t)h_t]^{1-\alpha} - c_t, \gamma u_t h_t; t + 1),$$

where $v(k_t, h_t; t)$ is the value function for period t and the constraints (1) and (2), which will always bind at the optimum, have been substituted into the value function $v(k_{t+1}, h_{t+1}; t + 1)$ for period $t + 1$. To begin characterizing the solution to this model, guess that the value function takes the time-invariant form

$$v(k_t, h_t; t) = v(k_t, h_t) = E + F \ln(k_t) + G \ln(h_t),$$

where E, F, and G are constants to be determined. Then, rewrite the Bellman equation as it appears using this guess.

2. For this problem, the optimality conditions include two first-order conditions, one for the optimal choice of c_t and the other for the optimal choice of u_t, and two envelope conditions, one for k_t and the other for h_t. Write down these first-order and envelope conditions.

3. Together with the binding constraints (1) and (2), your four equations from above form a system of six equations in six unknowns: the unknown variables c_t, u_t, k_t, and h_t and the unknown constants F and G. Once all these unknowns have been solved for, they can be substituted back into the Bellman equation itself to solve for E. As the next step in characterizing the model's solution, use the first-order and envelope conditions to solve for the unknown constants F and G in terms of the model's parameters α, β, and γ.

4. Now, substitute these solutions for F and G back into the first-order conditions for c_t and u_t to obtain expressions that show how the optimal choices for these flow variables depend on the model's parameters as well as the beginning-of-period stock variables k_t and h_t.

5. Substitute your solutions for c_t and u_t into the binding constraints to obtain a system of two difference equations that show how the stock variables k_{t+1} and h_{t+1} at time $t + 1$ get determined as functions of the stock variables k_t and h_t at time t.

6. What does the difference equation for h_t that you just derived tell you about the growth rate of the human capital stock in this model? How does that growth rate depend on the parameters α, β, and γ?

7. For the sake of completeness, substitute the results you've just derived back into the Bellman equation to find the solution for the remaining unknown constant E in terms of the parameters α, β, and γ.

If you have the time and are interested, another thing you can do is to put the difference equations describing the evolution of k_t and h_t and the equations determining the optimal choices of c_t and u_t into a computer program and see how all four of these variables evolve after you assign specific numerical values to the parameters α, β, and γ and to the initial stocks k_0 and h_0. If you do this, and set γ so that $\beta\gamma > 1$, what you should find is that u_t remains constant while the growth rates of the remaining three variables, k_t, h_t and c_t, all converge to the the same constant rate as t grows larger. Unlike the simpler neoclassical growth model, in which k_t and c_t converge to steady-state values, the Uzawa-Lucas model features sustained growth even in the long run. In terms of the models' structure, these contrasting results hold mainly because (i) in both models, there are decreasing marginal returns to the accumulation of physical capital but (ii) in the Lucas-Uzawa model, those same decreasing returns do not apply to the accumulation of human capital, thanks to the linearity of the human capital accumulation constraint (1).