Problem Set 11

ECON 772001 - Math for Economists | Peter Ireland
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Due Thursday, November 8

1. Natural Resource Depletion

Consider a continuous time version of the natural resource depletion problem that you solved previously, in discrete time, using the Kuhn-Tucker theorem in problem set 6. Let $c(t)$ denote society's consumption of an exhaustible resource during each period $t \in [0, \infty)$, and suppose that a representative consumer gets utility from this resource as described by

$$\int_0^\infty e^{-\rho t} \ln(c(t))\, \mathrm{d}t, \tag{1}$$

where the discount rate satisfies $\rho > 0$. Let $s(t)$ denote the stock of the resource that remains at each date $t \in [0, \infty)$. Since the resource is nonrenewable, this stock evolves according to

$$-c(t) \geq \dot{s}(t) \tag{2}$$

for $t \in [0, \infty)$, which just indicates that consumption during period t subtracts from the stock that remains to be consumed from that period forward. Now the social planner's problem can be stated as: choose continuously differentiable functions $c(t)$ and $s(t)$ for $t \in [0, \infty)$ to maximize (1) subject to $s(0) = s_0$ given and (2) for all $t \in [0, \infty)$.

a. Define (write down) the maximized Hamiltonian for this problem, using $\pi(t)$ to denote the multiplier corresponding to the constraint (2) for period t.

b. Next, write down the first-order condition for $c(t)$ and the pair of differential equations for $\pi(t)$ and $s(t)$ that, according to the maximum principle, help characterize the solution to the social planner's problem.

c. Probably the easiest way to solve the pair of differential equations that you just derived in part (b) from above is to use a guess-and-verify method. Accordingly, guess that those differential equations have solutions for the form

$$\pi(t) = \pi$$

and

$$s(t) = \left(\frac{1}{\pi\rho}\right) e^{-\rho t} + k,$$

where π and k are two constants that remain to be determined, and verify that these two functions do in fact satisfy the differential equations that you derived in part (b) above.

d. The maximum principle also gives us two boundary conditions that must be satisfied by the functions $\pi(t)$ and $s(t)$: the initial condition

$$s(0) = s_0 \text{ given}$$

and the terminal, or transversality, condition

$$\lim_{T\to\infty} \pi(T)s(T) = 0.$$

Use these boundary conditions to find specific values for the two unknown constants π and k, expressed in terms of the parameters ρ and s_0, that enter into the "general solutions" for $\pi(t)$ and $s(t)$ that you guessed and verified in part (c) above.

e. Finally, combine your "specific solutions" to the differential equations and the boundary conditions with the first-order condition for $c(t)$ to obtain an expression that shows how the optimal choice of $c(t)$ depends on the initial resource stock s_0 and the discount rate ρ. Is consumption rising, falling, or staying constant over time?

2. Investment with Adjustment Costs

Another "canonical" model that often gets analyzed using the maximum principle is the one solved by a firm that faces costs of adjusting its capital stock. During each period $t \in [0, \infty)$, this firm produces a flow of $Y(t)$ units of output using a stock of $K(t)$ units of capital according to the production function

$$Y(t) = K(t)^\alpha,$$

with $0 < \alpha < 1$. The capital stock depreciates at the constant rate δ, where $0 < \delta < 1$; hence, by investing $I(t)$ units of output at time t, the firm augments its capital stock according to

$$I(t) - \delta K(t) \geq \dot{K}(t). \tag{3}$$

At the same time, however, the firm incurs a quadratic cost of adjustment given by

$$(\phi/2)I(t)^2,$$

where the parameter $\phi > 0$ governs the magnitude of this cost. In this problem, the firm's choice of $I(t)$ can be positive, in which case the firm is installing new capital, or negative, in which case the firm is selling off existing capital; but either way, the formulation implies that it will incur adjustment costs in making these changes. The firm sells off whatever output remains after its investment choice is made; its profits during period t are therefore

$$K(t)^\alpha - I(t) - (\phi/2)I(t)^2.$$

Finally, let $r > 0$ denote the constant discount rate, reflecting either impatience on the part of the firm's owners or a positive rate of interest at which profits received today can generate additional income if saved for the future.

The firm's problem can now be stated as: choose continuously differentiable functions $I(t)$ and $K(t)$ for $t \in [0, \infty)$ to maximize the discounted value of profits over the infinite horizon,

$$\int_0^\infty e^{-rt}[K(t)^\alpha - I(t) - (\phi/2)I(t)^2]\,\mathrm{d}t,$$

subject to the capital accumulation constraint (3) for all $t \in [0, \infty)$, taking the initial capital stock $K(0) > 0$ as given.

a. Define (write down) the maximized Hamiltonian for this problem.

b. Now write down the first-order condition and the pair of differential equations that, according to the maximum principle, are necessary conditions for the values of $I(t)$ and $K(t)$ that solve the firm's infinite-horizon problem.

c. Your results from part (b) above should take the firm of a system of three equations in three unknowns. Use one of these equations to eliminate one of the variables – the additional variable that you introduced into the problem in defining the Hamiltonian and that corresponds to the Lagrange multiplier on (3) that would appear in the optimality conditions if you had derived them using the Kuhn-Tucker theorem instead – to rewrite the system as one involving two equations in two unknowns: investment $I(t)$ and the capital stock $K(t)$.

d. Using your results from part (c) above, write down a set of two equations that determine the optimal steady-state values I^* for investment and K^* for the capital stock (*Note:* it will not be possible to actually solve these equations for I^* and K^* algebraically; if you want to find these steady-state values you would have to assign numerical values to each of the model's parameters and then use a calculator or a computer to find the corresponding numerical values of I^* and K^*).

e. Finally, use your results from above to draw a phase diagram that illustrates the following property of the solution to the firm's problem: starting from any value $K(0) > 0$ for the initial capital stock, there is a unique value of investment $I(0)$ such that, starting from $I(0)$ and $K(0)$, the optimally-chosen paths for $I(t)$ and $K(t)$ converge to the steady-state values I^* and K^*. In drawing this phase diagram, it may be helpful to note that while investment $I(t)$ can take on positive or negative values, depending on whether the firm is accumulating or selling off capital, the capital stock $K(t)$ itself must always remain positive when these variables are chosen optimally.