Problem Set 10

ECON 772001 - Math for Economists — Peter Ireland
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Due Tuesday, November 10

This problem asks you to derive some of the implications of a model of stock prices and consumption due originally to Robert Lucas, "Asset Prices in an Exchange Economy," *Econometrica*, November 1978, pp.1429-1445.

Consider an economy populated by a large number of identical consumers, each of whom has preferences described by the additively-time separable utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1}$$

where the discount factor satisfies $0 < \beta < 1$ and the single-period utility function u is strictly increasing, strictly concave, and satisfies $\lim_{c \to 0} u'(c) = \infty$, this last assumption allowing us to ignore nonnegativity constraints on consumption in all of the analysis that follows, knowing in advance that they will never bind.

Each consumer finances his or her consumption by trading equity shares in the economy's productive assets: let's call them "fruit trees." Each share in each tree provides a dividend in the form of d_t pieces of "fruit" during each period $t = 0, 1, 2, ...$, where fruit is the economy's only consumption good. Let s_t denote the number of shares carried by a representative consumer into each period $t = 0, 1, 2, ...$, and let p_t denote the price of each share in each tree during each period $t = 0, 1, 2, ...$

Then, as sources of funds during each period $t = 0, 1, 2, ...$, the representative consumer has his or her dividend payments $d_t s_t$ and the total value $p_t s_t$ of the shares carried into the period. And as uses of funds during each period $t = 0, 1, 2, ...$, the consumer has his or her consumption c_t and the value $p_t s_{t+1}$ of the shares that he or she will carry into period $t + 1$. The representative consumer therefore faces the budget constraint

$$(d_t + p_t)s_t \geq c_t + p_t s_{t+1}$$

for all $t = 0, 1, 2, ...$ For the analysis that follows, it is useful, though not essential, to divide both sides of this budget constraint through by p_t and rearrange the resulting terms to write

$$\frac{d_t s_t - c_t}{p_t} \geq s_{t+1} - s_t. \tag{2}$$

These manipulations don't change the economic meaning of the constraint; instead, they just work to put the constraint in the same form as the one governing the evolution of the stock variable s_t that appeared in the general problem that we used to state and prove our version of the maximum principle for the discrete-time case.

1. The Kuhn-Tucker Formulation

The representative consumer therefore takes s_0 as given, and chooses sequences $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ to maximize the utility function in (1) subject to the constraint in (2), which as noted above must hold for all periods $t = 0, 1, 2, ...$ To begin the analysis, set up the Lagrangian for this infinite-horizon problem and write down the first-order conditions that characterize the consumer's optimal choices. Together with the binding constraint, these first-order conditions should form a system of three equations in three unknowns: the values of c_t and s_t that solve the consumer's problem together with the value of the Lagrange multiplier on the budget constraint.

2. The Maximum Principle

Now write down the maximized Hamiltonian for the consumer's problem, and use the maximum principle to rederive the same three optimality conditions that you obtained above.

3. An Economic Interpretation of the Results

One problem with these optimality conditions, regardless of whether they are derived using the method of Lagrange multipliers or the maximum principle, is that they make reference to a variable, the multiplier on the budget constraint, that lacks an immediate economic interpretation. To sidestep this problem, consider that a share costing p_{t-1} during period $t-1$ yields a dividend d_t during period t and can then be sold for p_t during period t. The total return on this share, accounting for both the dividend payment and the capital gain (or loss, if the price of the share declines instead of rising between $t-1$ and t), can therefore be measured as

$$R_t^s = \frac{d_t + p_t}{p_{t-1}}.$$

Combine the first-order conditions that you obtained through either method, above, to eliminate their references to the unknown multiplier. Your result should link the consumer's intertemporal marginal rate of substitution, $\beta u'(c_t)/u'(c_{t-1})$ to the asset return R_t^s in a way that is similar to how, in a static environment, the solution to a consumer's problem links the marginal rate of substitution between two goods to the relative price of those same two goods.

4. An Equilibrium Asset-Pricing Formula

Note that throughout the analysis so far, we've neglected to impose a nonnegativity constraint on the consumer's share holdings s_t. Negative values for s_t, however, can be interpreted meaningfully as describing situations where the consumer has taken a "short" position in shares: borrowing shares today that must be returned at some future date. In a finite-horizon version of this consumer's problem, it would make sense to impose a nonnegativity constraint on the terminal value s_{T+1}, interpreted as the consumer's share holdings at the end of the last period T, in order to require the consumer to repay all that he or she has borrowed in earlier periods. The transversality condition for the finite-horizon problem would then imply that $s_{T+1} = 0$ since, otherwise, the consumer would be "overaccumulating" shares and could enjoy higher utility by consuming more instead. By extension, the transversality condition that, loosely speaking, rules out both "borrowing without ever repaying" and "overaccumulating shares" in the infinite horizon case is one that requires

$$\lim_{T\to\infty} \pi_{T+1}s_{T+1} = 0,$$

where π_{T+1} is the Lagrange multiplier on the budget constraint linking periods T and $T+1$ (if you used a different notation for this multiplier in your own work above, that's no problem, you'll just have to adjust your own statement of this transversality condition accordingly to match your own notation for the multiplier).

Suppose now that there is one share per tree and one tree per consumer in the economy as whole. Then, in equilibrium, it must be that $s_t = 1$ for all $t = 0, 1, 2, ...$ In this case, the representative consumer's budget constraint (2) implies that

$$c_t = d_t \tag{3}$$

must also hold in equilibrium for all $t = 0, 1, 2, ...$ In words, (3) just says that with one tree per consumer in the economy, prices must adjust in equilibrium so that the representative consumer ends up eating the fruit from his or her tree. In equilibrium, the transversality condition also requires something more specific, namely, that

$$\lim_{T\to\infty} \pi_{T+1} = 0. \tag{4}$$

Go back to the optimality conditions that you derived in answering question 1 or 2 above, and using (3) and (4) as well, show that in equilibrium, the relationship

$$p_t = \sum_{j=1}^{\infty} \left[\frac{\beta^j u'(d_{t+j})}{u'(d_t)} \right] d_{t+j}$$

must also hold. This last equation expresses a familiar condition from financial economics – that the price of a share of stock equals the present discount value of the future dividends paid by that share – but adds an element from macroeconomics by requiring that the discounting uses the representative consumer's marginal rate of intertemporal substitution evaluated using the goods market-clearing condition (3) for all $t = 0, 1, 2, ...$

5. An Interesting Special Case

Suppose now that the representative consumer's single-period utility takes the (natural) logarithmic form

$$u(c_t) = \ln(c_t).$$

Use you results from above to show that, in this case, the theory predicts that the "dividend yield" d_t/p_t on shares will be constant over time, despite the fact that the dividends d_t and the stock price p_t themselves can fluctuate.