Problem Set 1

ECON 772001 - Math for Economists | Peter Ireland
Boston College, Department of Economics | Fall 2020

Due Tuesday, September 15

1. Profit Maximization

Consider a firm that produces output y with capital k and labor l according to the technology described by

$$k^a l^b \geq y, \tag{1}$$

where $0 < a < 1$, $0 < b < 1$, and $0 < a + b < 1$. The firm sells each unit of output at the price p, rents each unit of capital at the rate r, and hires each unit of labor at the wage w. Hence it chooses y, k, and l to maximize profits

$$py - rk - wl$$

subject to the constraint just shown in (1).

a. Set up the Lagrangian for this problem, letting λ denote the multiplier on the constraint.

b. Next, write down the conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values y^*, k^*, and l^* that solve the firm's problem, together with the associated value λ^* for the multiplier.

c. Assume that the constraint binds at the optimum (can you tell under what conditions this will be true?), and use your results from above to solve for y^*, k^*, l^*, and λ^* in terms of the model's parameters: a, b, p, r, and w.

d. Finally, use your solutions from above to answer the following questions:

   i. What happens to the optimal y^*, k^*, and l^* when the output price p rises, holding all other parameters fixed? In each case, does the optimal choice rise, fall, or stay the same?

   ii. What happens to the optimal y^*, k^*, and l^* when the rental rate for capital r rises, holding all other parameters fixed?

   iii. What happens to the optimal y^*, k^*, and l^* when the wage rate w rises, holding all other parameters fixed?

   iv. What happens to the optimal y^*, k^*, and l^* when p, r, and w all double at the same time?

2. Utility Maximization

Now consider a consumer who uses his or her income I to purchase c_1 units of good 1 at the price of p_1 per unit and c_2 units of good 2 at the price of p_2 per unit, subject to the budget constraint

$$I \geq p_1 c_1 + p_2 c_2. \tag{2}$$

Suppose that the consumer has preferences over the two goods described by the utility function

$$U(c_1, c_2) = c_1^a c_2^{1-a}, \tag{3}$$

where $0 < a < 1$.

a. Set up the Lagrangian for the consumer's problem: choose c_1 and c_2 to maximize utility given in (3) subject to the budget constraint shown in (2), letting λ denote the multiplier on the constraint.

b. Next, write down the conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values c_1^* and c_2^* that solve the consumer's problem, together with the associated value λ^* for the multiplier.

c. Assume that the budget constraint binds at the optimum (again, can you tell under what conditions this will be true?), and use your results from above to solve for c_1^*, c_2^*, and λ^* in terms of the model's parameters: I, p_1, p_2, and a.

d. Finally, use your answers from above to answer the following questions. What is the relationship between the preference parameter a and the fraction $p_1 c_1^*/I$ of income that the consumer optimally spends on good one? And what is the relationship between $1 - a$ and the fraction $p_2 c_2^*/I$ of income that the consumer optimally spends on good two?

3. Utility Maximization (Again)

Redo the four parts of the previous question, but assuming that instead of (3), the consumer's utility is described by

$$U(c_1, c_2) = a \ln(c_1) + (1 - a) \ln(c_2),$$

where ln denotes the natural logarithm and where $0 < a < 1$ as before.