Solutions to Problem Set 9

ECON 772001 - Math for Economists — Peter Ireland
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1. Natural Resource Depletion

The social planner chooses sequences $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to the constraints

$$s_0 \text{ given}$$

and

$$s_t - c_t \geq s_{t+1}$$

for all $t = 0, 1, 2, ...$

a. With the Lagrangian for this problem defined as

$$L = \sum_{t=0}^{\infty} \beta^t \ln(c_t) + \sum_{t=0}^{\infty} \pi_{t+1}(s_t - c_t - s_{t+1}),$$

the first-order condition for the optimal choice of c_t can be written as

$$\frac{\beta^t}{c_t} - \pi_{t+1} = 0$$

and the first-order condition for the optimal choice of s_t can be written as

$$\pi_{t+1} - \pi_t = 0.$$

b. Since the first order condition for c_t must hold for all $t = 0, 1, 2, ...$, it can be rolled back one period to obtain

$$\frac{\beta^{t-1}}{c_{t-1}} - \pi_t = 0.$$

Hence, the first-order condition for s_t implies that

$$\frac{\beta^t}{c_t} = \frac{\beta^{t-1}}{c_{t-1}}$$

or more simply that

$$c_t = \beta c_{t-1},$$

which reveals that it is optimal to have c_t declining over time.

c. The transversality condition for this infinite horizon problem is

$$\lim_{T\to\infty} \pi_{T+1} s_{T+1} = 0.$$

Since the first-order conditions for c_t and s_t imply that π_{t+1} must equal some positive constant, however, this transversality condition requires, more simply that,

$$\lim_{T\to\infty} s_{T+1} = 0.$$

As shown in answering part (b), the optimal path for consumption must satisfy

$$c_t = \beta c_{t-1}$$

or

$$c_t = \beta^t c_0$$

for all $t = 0, 1, 2, \ldots$. Meanwhile, the binding constraint implies that

$$s_1 = s_0 - c_0,$$

$$s_2 = s_1 - c_1 = s_0 - c_0 - c_1,$$

and, more generally,

$$s_{T+1} = s_0 - \sum_{t=0}^{T} c_t = s_0 - c_0 \sum_{t=0}^{T} \beta^t.$$

Taking the limits on both sides of this last equation as $T \to \infty$ and using the transversality condition yields

$$0 = s_0 - c_0 \sum_{t=0}^{\infty} \beta^t.$$

Note, finally, that

$$\sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta}.$$

The easiest way to see this is to multiply both sides of equation by $1 - \beta$ and then write out the terms in the infinite sum:

$$(1-\beta) \sum_{t=0}^{\infty} \beta^t = (1 + \beta + \beta^2 + \ldots) - (\beta + \beta^2 + \beta^3 + \ldots) = 1.$$

Thus, $c_0 = (1-\beta)s_0$, implying that the optimal path for consumption has

$$c_t = \beta^t (1-\beta) s_0$$

for all $t = 0, 1, 2, \ldots$.

d. To re-derive the optimality conditions from part (a) using the maximum principle instead of the method of Lagrange multipliers, define the maximized present value Hamiltonian as

$$H(s_t, \pi_{t+1}; t) = \max_{c_t} \beta^t \ln(c_t) - \pi_{t+1} c_t.$$

The maximum principle then implies that the solution to the dynamic optimization problem must satisfy the first-order condition

$$\frac{\beta^t}{c_t} - \pi_{t+1} = 0$$

and the pair of difference equations

$$\pi_{t+1} - \pi_t = -H_s(s_t, \pi_{t+1}; t) = 0$$

and

$$s_{t+1} - s_t = H_\pi(s_t, \pi_{t+1}; t) = -c_t.$$

The first two of these optimality conditions reproduce exactly the first-order conditions for c_t and s_t from part (a), while the third is simply a restatement of the constraint.

2. Life Cycle Saving

The consumer chooses sequences $\{c_t\}_{t=0}^{T}$ and $\{k_t\}_{t=1}^{T+1}$ to maximize

$$\sum_{t=0}^{T} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right)$$

subject to the constraints

$$k_0 = 0 \text{ given,}$$

$$w_t + r_t k_t - c_t \geq k_{t+1} - k_t$$

for all $t = 0, 1, ..., T$, and

$$k_{T+1} \geq k^* > 0.$$

a. The maximized Hamiltonian for the consumer's problem can be defined as

$$H(k_t, \pi_{t+1}; t) = \max_{c_t} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right) + \pi_{t+1}(w_t + r_t k_t - c_t).$$

b. According to the maximum principle, the solution to the consumer's dynamic optimization problem is characterized by the first-order condition

$$\beta^t c_t^{-\sigma} - \pi_{t+1} = 0,$$

the pair of difference equations

$$\pi_{t+1} - \pi_t = -H_k(k_t, \pi_{t+1}; t) = -\pi_{t+1} r_t$$

and

$$k_{t+1} - k_t = H_\pi(k_t, \pi_{t+1}; t) = w_t + r_t k_t - c_t,$$

the initial condition

$$k_0 = 0,$$

and the terminal or transversality condition

$$\pi_{T+1}(k_{T+1} - k^*) = 0.$$

c. Since the first-order condition for c_t must hold for all periods $= 0, 1, ...T$, it implies that

$$\pi_{t+1} = \beta^t c_t^{-\sigma}$$

and

$$\pi_t = \beta^{t-1} c_{t-1}^{-\sigma}.$$

Substituting these conditions into the difference equation for π_t yields

$$(1 + r_t)\beta^t c_t^{-\sigma} = \beta^{t-1} c_{t-1}^{-\sigma}$$

or

$$c_t / c_{t-1} = [\beta(1 + r_t)]^{1/\sigma}.$$