

## Solutions to Problem Set 8

ECON 772001 - Math for Economists  
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For Extra Practice - Not Collected or Graded

### 1. Optimal Growth

The representative consumer or social planner chooses functions  $c(t)$  for  $t \in [0, \infty)$  and  $k(t)$  for  $t \in (0, \infty)$  to maximize

$$\int_0^{\infty} e^{-\rho t} \left[ \frac{c(t)^{1-\sigma} - 1}{1-\sigma} \right] dt$$

subject to

$$k(0) \text{ given}$$

and

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all  $t \in [0, \infty)$ .

a. The maximized current value Hamiltonian for the consumer's problem is

$$H(k(t), \theta(t)) = \max_{c(t)} \left[ \frac{c(t)^{1-\sigma} - 1}{1-\sigma} \right] + \theta(t)[k(t)^\alpha - \delta k(t) - c(t)].$$

b. According to the maximum principle, the solution to the consumer's dynamic optimization problem is characterized by the first-order condition

$$c(t)^{-\sigma} - \theta(t) = 0,$$

the pair of differential equations

$$\dot{\theta}(t) = \rho\theta(t) - H_k(k(t), \theta(t); t) = \rho\theta(t) - \theta(t)[\alpha k(t)^{\alpha-1} - \delta]$$

and

$$\dot{k}(t) = H_\theta(k(t), \theta(t); t) = k(t)^\alpha - \delta k(t) - c(t),$$

where a unique solution to the system is pinned down by the initial condition

$$k(0) \text{ given,}$$

and the terminal or transversality condition

$$\lim_{T \rightarrow \infty} e^{-\rho T} \theta(T) k(T) = \lim_{T \rightarrow \infty} e^{-\rho T} c(T)^{-\sigma} k(T) = 0.$$

c. Rewrite the first-order condition for  $c(t)$  as

$$1 = \theta(t)c(t)^\sigma$$

and differentiate both sides with respect to  $t$  to get

$$0 = \dot{\theta}(t)c(t)^\sigma + \sigma\theta(t)c(t)^{\sigma-1}\dot{c}(t).$$

Now use the differential equation for  $\theta(t)$ , to eliminate  $\dot{\theta}(t)$  from this last expression to obtain

$$0 = \rho\theta(t)c(t)^\sigma - \theta(t)[\alpha k(t)^{\alpha-1} - \delta]c(t)^\sigma + \sigma\theta(t)c(t)^{\sigma-1}\dot{c}(t).$$

Divide this result through by  $\sigma\theta(t)c(t)^{\sigma-1}$  to obtain

$$\dot{c}(t) = (1/\sigma)[\alpha k(t)^{\alpha-1} - \delta - \rho]c(t),$$

which can be combined with the constraint

$$\dot{k}(t) = k(t)^\alpha - \delta k(t) - c(t)$$

to obtain a system of two differential equations in the two unknown functions  $c(t)$  and  $k(t)$ .

d. The differential equation for  $c(t)$  implies that  $\dot{c}(t) = 0$  when  $k(t) = k^*$ ,  $\dot{c}(t) > 0$  when  $k(t) < k^*$ , and  $\dot{c}(t) < 0$  when  $k(t) > k^*$ , where

$$k^* = \left(\frac{\delta + \rho}{\alpha}\right)^{1/(\alpha-1)}.$$

The differential equation for  $k(t)$  implies that  $\dot{k}(t) = 0$  when  $c(t) = k(t)^\alpha - \delta k(t)$ ,  $\dot{k}(t) > 0$  when  $c(t) < k(t)^\alpha - \delta k(t)$ , and  $\dot{k}(t) < 0$  when  $c(t) > k(t)^\alpha - \delta k(t)$ . All of these properties are the same, regardless of the value for the coefficient of relative risk aversion  $\sigma$ , so the phase diagram looks exactly the same, with the specific value assigned to  $\sigma$  affecting only the speed with which the economy converges to its steady state.