Solutions to Problem Set 8

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2020

For Extra Practice - Not Collected or Graded

1. Optimal Growth

The representative consumer or social planner chooses functions $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ to maximize

$$\int_0^\infty e^{-\rho t} \left[\frac{c(t)^{1-\sigma} - 1}{1-\sigma} \right] \mathrm{d}t$$

subject to

$$k(0) \text{ given}$$

and

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$.

a. The maximized current value Hamiltonian for the consumer's problem is

$$H(k(t), \theta(t)) = \max_{c(t)} \left[\frac{c(t)^{1-\sigma} - 1}{1-\sigma} \right] + \theta(t)[k(t)^\alpha - \delta k(t) - c(t)].$$

b. According to the maximum principle, the solution to the consumer's dynamic optimization problem is characterized by the first-order condition

$$c(t)^{-\sigma} - \theta(t) = 0,$$

the pair of differential equations

$$\dot{\theta}(t) = \rho\theta(t) - H_k(k(t), \theta(t); t) = \rho\theta(t) - \theta(t)[\alpha k(t)^{\alpha-1} - \delta]$$

and

$$\dot{k}(t) = H_\theta(k(t), \theta(t); t) = k(t)^\alpha - \delta k(t) - c(t),$$

where a unique solution to the system is pinned down by the initial condtion

$$k(0) \text{ given},$$

and the terminal or transversality condition

$$\lim_{T\to\infty} e^{-\rho T}\theta(T)k(T) = \lim_{T\to\infty} e^{-\rho T}c(T)^{-\sigma}k(T) = 0.$$

c. Rewrite the first-order condition for $c(t)$ as

$$1 = \theta(t)c(t)^{\sigma}$$

and differentiate both sides with respect to t to get

$$0 = \dot{\theta}(t)c(t)^{\sigma} + \sigma\theta(t)c(t)^{\sigma-1}\dot{c}(t).$$

Now use the differential equation for $\theta(t)$, to eliminate $\dot{\theta}(t)$ from this last expression to obtain

$$0 = \rho\theta(t)c(t)^{\sigma} - \theta(t)[\alpha k(t)^{\alpha-1} - \delta]c(t)^{\sigma} + \sigma\theta(t)c(t)^{\sigma-1}\dot{c}(t).$$

Divide this result through by $\sigma\theta(t)c(t)^{\sigma-1}$ to obtain

$$\dot{c}(t) = (1/\sigma)[\alpha k(t)^{\alpha-1} - \delta - \rho]c(t),$$

which can be combined with the constraint

$$\dot{k}(t) = k(t)^{\alpha} - \delta k(t) - c(t)$$

to obtain a system of two differential equations in the two unknown functions $c(t)$ and $k(t)$.

d. The differential equation for $c(t)$ implies that $\dot{c}(t) = 0$ when $k(t) = k^*$, $\dot{c}(t) > 0$ when $k(t) < k^*$, and $\dot{c}(t) < 0$ when $k(t) > k^*$, where

$$k^* = \left(\frac{\delta + \rho}{\alpha}\right)^{1/(\alpha-1)}.$$

The differential equation for $k(t)$ implies that $\dot{k}(t) = 0$ when $c(t) = k(t)^{\alpha} - \delta k(t)$, $\dot{k}(t) > 0$ when $c(t) < k(t)^{\alpha} - \delta k(t)$, and $\dot{k}(t) < 0$ when $c(t) > k(t)^{\alpha} - \delta k(t)$. All of these properties are the same, regardless of the value for the coefficient of relative risk aversion σ, so the phase diagram looks exactly the same, with the specific value assigned to σ affecting only the speed with which the economy converges to its steady state.