

## Solutions to Problem Set 7

ECON 772001 - Math for Economists  
Boston College, Department of Economics

Peter Ireland  
Fall 2018

For Extra Practice – Not Collected or Graded

### 1. The Permanent Income Hypothesis

The consumer chooses  $c_0$ ,  $c_1$ , and  $s$  to maximize the utility function

$$\ln(c_0) + \beta \ln(c_1)$$

subject to the constraints

$$w_0 \geq c_0 + s$$

and

$$w_1 + (1 + r)s \geq c_1.$$

a. With the Lagrangian for the consumer's problem defined as

$$L(c_0, c_1, s, \lambda_0, \lambda_1) = \ln(c_0) + \beta \ln(c_1) + \lambda_0(w_0 - c_0 - s) + \lambda_1[w_1 + (1 + r)s - c_1],$$

the first-order conditions can be written as

$$\frac{1}{c_0^*} - \lambda_0^* = 0,$$

$$\frac{\beta}{c_1^*} - \lambda_1^* = 0,$$

and

$$-\lambda_0^* + \lambda_1^*(1 + r) = 0$$

and the constraints, which will bind at the optimum, can be written as

$$w_0 - c_0^* - s^* = 0$$

and

$$w_1 + (1 + r)s^* - c_1^* = 0.$$

Combine the budget constraints to obtain

$$w_0 + \frac{w_1}{1 + r} = c_0^* + \frac{c_1^*}{1 + r},$$

which says that the present value of the consumer's income will equal the present value of his or her consumption over the two periods. Now combine the first-order conditions to obtain

$$\frac{1}{c_0^*} = \frac{\beta(1 + r)}{c_1^*}$$

or

$$c_1^* = \beta(1+r)c_0^*.$$

Substitute this last expression into the present value budget constraint to find the desired solution

$$c_0^* = \frac{1}{1+\beta} \left( w_0 + \frac{w_1}{1+r} \right),$$

then substitute this solution into the previous expression to obtain

$$c_1^* = \frac{\beta(1+r)}{1+\beta} \left( w_0 + \frac{w_1}{1+r} \right).$$

Finally, use the budget constraint for period  $t = 0$  to obtain

$$s^* = w_0 - \frac{1}{1+\beta} \left( w_0 + \frac{w_1}{1+r} \right).$$

- b. When  $\beta(1+r) = 1$ , the solutions from above simplify to imply that consumption remains constant over time, with

$$c_0^* = c_1^* = \frac{1}{1+\beta} \left( w_0 + \frac{w_1}{1+r} \right).$$

- c. When  $\beta(1+r) = 1$ , the solution for saving becomes

$$s^* = \left( \frac{\beta}{1+\beta} \right) (w_0 - w_1),$$

which implies that the consumer borrows or saves during period  $t = 0$  depending on whether  $w_1$  is greater than or less than  $w_0$ .

## 2. Habit Formation

With preferences modified to reflect habit formation, the consumer chooses  $c_0$ ,  $c_1$ , and  $s$  to maximize the utility function

$$\ln(c_0) + \beta \ln(c_1 - \gamma c_0)$$

subject to the constraints

$$w_0 \geq c_0 + s$$

and

$$w_1 + (1+r)s \geq c_1.$$

- a. With the Lagrangian for the consumer's problem defined as

$$L(c_0, c_1, s, \lambda_0, \lambda_1) = \ln(c_0) + \beta \ln(c_1 - \gamma c_0) + \lambda_0(w_0 - c_0 - s) + \lambda_1[w_1 + (1+r)s - c_1],$$

the first-order conditions can be written as

$$\frac{1}{c_0^*} - \frac{\beta\gamma}{c_1^* - \gamma c_0^*} - \lambda_0^* = 0,$$

$$\frac{\beta}{c_1^* - \gamma c_0^*} - \lambda_1^* = 0,$$

and

$$-\lambda_0^* + \lambda_1^*(1+r) = 0$$

and the constraints, which will bind at the optimum, can be written as

$$w_0 - c_0^* - s^* = 0$$

and

$$w_1 + (1+r)s^* - c_1^* = 0.$$

Combine the budget constraints to obtain

$$w_0 + \frac{w_1}{1+r} = c_0^* + \frac{c_1^*}{1+r},$$

as before. Now combine the first-order conditions to obtain

$$\frac{1}{c_0^*} - \frac{\beta\gamma}{c_1^* - \gamma c_0^*} = \frac{\beta(1+r)}{c_1^* - \gamma c_0^*}$$

or

$$c_1^* = [(1+\beta)\gamma + \beta(1+r)]c_0^*.$$

Substitute this last expression into the present value budget constraint to find the desired solution

$$c_0^* = \left[ \frac{1+r}{(1+\beta)(1+r+\gamma)} \right] \left( w_0 + \frac{w_1}{1+r} \right),$$

then substitute this solution into the previous expression to obtain

$$c_1^* = \left\{ \frac{(1+r)[(1+\beta)\gamma + \beta(1+r)]}{(1+\beta)(1+r+\gamma)} \right\} \left( w_0 + \frac{w_1}{1+r} \right).$$

Finally, use the budget constraint for period  $t = 0$  to obtain

$$s^* = w_0 - \left[ \frac{1+r}{(1+\beta)(1+r+\gamma)} \right] \left( w_0 + \frac{w_1}{1+r} \right).$$

b. When  $\beta(1+r) = 1$ , the solutions from above simplify to

$$c_0^* = \left[ \frac{1}{(1+\beta)(1+\beta\gamma)} \right] \left( w_0 + \frac{w_1}{1+r} \right)$$

and

$$c_1^* = \left[ \frac{(1+\beta)\gamma + 1}{(1+\beta)(1+\beta\gamma)} \right] \left( w_0 + \frac{w_1}{1+r} \right),$$

which imply that  $c_1^* > c_0^*$ .

### 3. Durable Consumption

With preferences modified to reflect durability in consumption, the consumer chooses  $c_0$ ,  $c_1$ , and  $s$  to maximize the utility function

$$\ln(c_0) + \beta \ln(c_1 + \theta c_0)$$

subject to the constraints

$$w_0 \geq c_0 + s$$

and

$$w_1 + (1 + r)s \geq c_1.$$

a. With the Lagrangian for the consumer's problem defined as

$$L(c_0, c_1, s, \lambda_0, \lambda_1) = \ln(c_0) + \beta \ln(c_1 + \theta c_0) + \lambda_0(w_0 - c_0 - s) + \lambda_1[w_1 + (1 + r)s - c_1],$$

the first-order conditions can be written as

$$\frac{1}{c_0^*} + \frac{\beta\theta}{c_1^* + \theta c_0^*} - \lambda_0^* = 0,$$

$$\frac{\beta}{c_1^* + \theta c_0^*} - \lambda_1^* = 0,$$

and

$$-\lambda_0^* + \lambda_1^*(1 + r) = 0$$

and the constraints, which will bind at the optimum, can be written as

$$w_0 - c_0^* - s^* = 0$$

and

$$w_1 + (1 + r)s^* - c_1^* = 0.$$

Combine the budget constraints to obtain

$$w_0 + \frac{w_1}{1 + r} = c_0^* + \frac{c_1^*}{1 + r},$$

as before. Now combine the first-order conditions to obtain

$$\frac{1}{c_0^*} + \frac{\beta\theta}{c_1^* + \theta c_0^*} = \frac{\beta(1 + r)}{c_1^* + \theta c_0^*}$$

or

$$c_1^* = [\beta(1 + r) - (1 + \beta)\theta]c_0^*.$$

Substitute this last expression into the present value budget constraint to find the desired solution

$$c_0^* = \left[ \frac{1 + r}{(1 + \beta)(1 + r - \theta)} \right] \left( w_0 + \frac{w_1}{1 + r} \right),$$

then substitute this solution into the previous expression to obtain

$$c_1^* = \left\{ \frac{(1+r)[\beta(1+r) - (1+\beta)\theta]}{(1+\beta)(1+r-\theta)} \right\} \left( w_0 + \frac{w_1}{1+r} \right).$$

Finally, use the budget constraint for period  $t = 0$  to obtain

$$s^* = w_0 - \left[ \frac{1+r}{(1+\beta)(1+r-\theta)} \right] \left( w_0 + \frac{w_1}{1+r} \right).$$

b. When  $\beta(1+r) = 1$ , the solutions from above simplify to

$$c_0^* = \left[ \frac{1}{(1+\beta)(1-\beta\theta)} \right] \left( w_0 + \frac{w_1}{1+r} \right)$$

and

$$c_1^* = \left[ \frac{1 - (1+\beta)\theta}{(1+\beta)(1-\beta\theta)} \right] \left( w_0 + \frac{w_1}{1+r} \right),$$

which imply that  $c_0^* > c_1^*$ .