Solutions to Problem Set 7

ECON 772001 - Math for Economists | Peter Ireland
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For Extra Practice – Not Collected or Graded

1. The Permanent Income Hypothesis

The consumer chooses c_0, c_1, and s to maximize the utility function

$$\ln(c_0) + \beta \ln(c_1)$$

subject to the constraints

$$w_0 \geq c_0 + s$$

and

$$w_1 + (1+r)s \geq c_1.$$

a. With the Lagrangian for the consumer's problem defined as

$$L(c_0, c_1, s, \lambda_0, \lambda_1) = \ln(c_0) + \beta \ln(c_1) + \lambda_0(w_0 - c_0 - s) + \lambda_1[w_1 + (1+r)s - c_1],$$

the first-order conditions can be written as

$$\frac{1}{c_0^*} - \lambda_0^* = 0,$$

$$\frac{\beta}{c_1^*} - \lambda_1^* = 0,$$

and

$$-\lambda_0^* + \lambda_1^*(1+r) = 0$$

and the constraints, which will bind at the optimum, can be written as

$$w_0 - c_0^* - s^* = 0$$

and

$$w_1 + (1+r)s^* - c_1^* = 0.$$

Combine the budget constraints to obtain

$$w_0 + \frac{w_1}{1+r} = c_0^* + \frac{c_1^*}{1+r},$$

which says that the present value of the consumer's income will equal the present value of his or her consumption over the two periods. Now combine the first-order conditions to obtain

$$\frac{1}{c_0^*} = \frac{\beta(1+r)}{c_1^*}$$

or

$$c_1^* = \beta(1+r)c_0^*.$$

Substitute this last expression into the present value budget constraint to find the desired solution

$$c_0^* = \frac{1}{1+\beta}\left(w_0 + \frac{w_1}{1+r}\right),$$

then substitute this solution into the previous expression to obtain

$$c_1^* = \frac{\beta(1+r)}{1+\beta}\left(w_0 + \frac{w_1}{1+r}\right).$$

Finally, use the budget constraint for period $t = 0$ to obtain

$$s^* = w_0 - \frac{1}{1+\beta}\left(w_0 + \frac{w_1}{1+r}\right).$$

b. When $\beta(1+r) = 1$, the solutions from above simplify to imply that consumption remains constant over time, with

$$c_0^* = c_1^* = \frac{1}{1+\beta}\left(w_0 + \frac{w_1}{1+r}\right).$$

c. When $\beta(1+r) = 1$, the solution for saving becomes

$$s^* = \left(\frac{\beta}{1+\beta}\right)(w_0 - w_1),$$

which implies that the consumer borrows or saves during period $t = 0$ depending on whether w_1 is greater than or less than w_0.

2. Habit Formation

With preferences modified to reflect habit formation, the consumer chooses c_0, c_1, and s to maximize the utility function

$$\ln(c_0) + \beta\ln(c_1 - \gamma c_0)$$

subject to the constraints

$$w_0 \geq c_0 + s$$

and

$$w_1 + (1+r)s \geq c_1.$$

a. With the Lagrangian for the consumer's problem defined as

$$L(c_0, c_1, s, \lambda_0, \lambda_1) = \ln(c_0) + \beta\ln(c_1 - \gamma c_0) + \lambda_0(w_0 - c_0 - s) + \lambda_1[w_1 + (1+r)s - c_1],$$

the first-order conditions can be written as

$$\frac{1}{c_0^*} - \frac{\beta\gamma}{c_1^* - \gamma c_0^*} - \lambda_0^* = 0,$$

$$\frac{\beta}{c_1^* - \gamma c_0^*} - \lambda_1^* = 0,$$

and

$$-\lambda_0^* + \lambda_1^*(1+r) = 0$$

and the constraints, which will bind at the optimum, can be written as

$$w_0 - c_0^* - s^* = 0$$

and

$$w_1 + (1+r)s^* - c_1^* = 0.$$

Combine the budget constraints to obtain

$$w_0 + \frac{w_1}{1+r} = c_0^* + \frac{c_1^*}{1+r},$$

as before. Now combine the first-order conditions to obtain

$$\frac{1}{c_0^*} - \frac{\beta\gamma}{c_1^* - \gamma c_0^*} = \frac{\beta(1+r)}{c_1^* - \gamma c_0^*}$$

or

$$c_1^* = [(1+\beta)\gamma + \beta(1+r)]c_0^*.$$

Substitute this last expression into the present value budget constraint to find the desired solution

$$c_0^* = \left[\frac{1+r}{(1+\beta)(1+r+\gamma)}\right]\left(w_0 + \frac{w_1}{1+r}\right),$$

then substitute this solution into the previous expression to obtain

$$c_1^* = \left\{\frac{(1+r)[(1+\beta)\gamma + \beta(1+r)]}{(1+\beta)(1+r+\gamma)}\right\}\left(w_0 + \frac{w_1}{1+r}\right).$$

Finally, use the budget constraint for period $t = 0$ to obtain

$$s^* = w_0 - \left[\frac{1+r}{(1+\beta)(1+r+\gamma)}\right]\left(w_0 + \frac{w_1}{1+r}\right).$$

b. When $\beta(1+r) = 1$, the solutions from above simplify to

$$c_0^* = \left[\frac{1}{(1+\beta)(1+\beta\gamma)}\right]\left(w_0 + \frac{w_1}{1+r}\right)$$

and

$$c_1^* = \left[\frac{(1+\beta)\gamma + 1}{(1+\beta)(1+\beta\gamma)}\right]\left(w_0 + \frac{w_1}{1+r}\right),$$

which imply that $c_1^* > c_0^*$.

3. Durable Consumption

With preferences modified to reflect durability in consumption, the consumer chooses c_0, c_1, and s to maximize the utility function

$$\ln(c_0) + \beta \ln(c_1 + \theta c_0)$$

subject to the constraints

$$w_0 \geq c_0 + s$$

and

$$w_1 + (1+r)s \geq c_1.$$

a. With the Lagrangian for the consumer's problem defined as

$$L(c_0, c_1, s, \lambda_0, \lambda_1) = \ln(c_0) + \beta \ln(c_1 + \theta c_0) + \lambda_0(w_0 - c_0 - s) + \lambda_1[w_1 + (1+r)s - c_1],$$

the first-order conditions can be written as

$$\frac{1}{c_0^*} + \frac{\beta\theta}{c_1^* + \theta c_0^*} - \lambda_0^* = 0,$$

$$\frac{\beta}{c_1^* + \theta c_0^*} - \lambda_1^* = 0,$$

and

$$-\lambda_0^* + \lambda_1^*(1+r) = 0$$

and the constraints, which will bind at the optimum, can be written as

$$w_0 - c_0^* - s^* = 0$$

and

$$w_1 + (1+r)s^* - c_1^* = 0.$$

Combine the budget constraints to obtain

$$w_0 + \frac{w_1}{1+r} = c_0^* + \frac{c_1^*}{1+r},$$

as before. Now combine the first-order conditions to obtain

$$\frac{1}{c_0^*} + \frac{\beta\theta}{c_1^* + \theta c_0^*} = \frac{\beta(1+r)}{c_1^* + \theta c_0^*}$$

or

$$c_1^* = [\beta(1+r) - (1+\beta)\theta]c_0^*.$$

Substitute this last expression into the present value budget constraint to find the desired solution

$$c_0^* = \left[\frac{1+r}{(1+\beta)(1+r-\theta)}\right]\left(w_0 + \frac{w_1}{1+r}\right),$$

then substitute this solution into the previous expression to obtain

$$c_1^* = \left\{ \frac{(1+r)[\beta(1+r) - (1+\beta)\theta]}{(1+\beta)(1+r-\theta)} \right\} \left(w_0 + \frac{w_1}{1+r} \right).$$

Finally, use the budget constraint for period $t = 0$ to obtain

$$s^* = w_0 - \left[\frac{1+r}{(1+\beta)(1+r-\theta)} \right] \left(w_0 + \frac{w_1}{1+r} \right).$$

b. When $\beta(1+r) = 1$, the solutions from above simplify to

$$c_0^* = \left[\frac{1}{(1+\beta)(1-\beta\theta)} \right] \left(w_0 + \frac{w_1}{1+r} \right)$$

and

$$c_1^* = \left[\frac{1-(1+\beta)\theta}{(1+\beta)(1-\beta\theta)} \right] \left(w_0 + \frac{w_1}{1+r} \right),$$

which imply that $c_0^* > c_1^*$.