Solutions to Problem Set 6

ECON 772001 - Math for Economists
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1. Optimal Lending

The lender's dynamic optimization problem can be stated formally as:

$$\max_{c_0^L, c_1^L, s^L} \ln(c_0^L) + \beta \ln(c_1^L) \text{ subject to } 1 \geq c_0^L + s^L \text{ and } (1+r)s^L \geq c_1^L.$$

a. The Lagrangian for this consumer's problem can be defined as

$$L(c_0^L, c_1^L, s^L, \lambda_0^L, \lambda_1^L) = \ln(c_0^L) + \beta \ln(c_1^L) + \lambda_0^L(1 - c_0^L - s^L) + \lambda_1^L[(1+r)s^L - c_1^L].$$

b. According to the Kuhn-Tucker theorem, the lender's optimal choices of c_0^{L*}, c_1^{L*}, and s^{L*}, together with the associated values of λ_0^{L*} and λ_1^{L*}, must satisfy the first-order conditions

$$L_1(c_0^{L*}, c_1^{L*}, s^{L*}, \lambda_0^{L*}, \lambda_1^{L*}) = 1/c_0^{L*} - \lambda_0^{L*} = 0,$$
$$L_2(c_0^{L*}, c_1^{L*}, s^{L*}, \lambda_0^{L*}, \lambda_1^{L*}) = \beta/c_1^{L*} - \lambda_1^{L*} = 0,$$

and

$$L_3(c_0^{L*}, c_1^{L*}, s^{L*}, \lambda_0^{L*}, \lambda_1^{L*}) = -\lambda_0^{L*} + \lambda_1^{L*}(1+r) = 0$$

the constraints

$$L_4(c_0^{L*}, c_1^{L*}, s^{L*}, \lambda_0^{L*}, \lambda_1^{L*}) = 1 - c_0^{L*} - s^{L*} \geq 0$$

and

$$L_5(c_0^{L*}, c_1^{L*}, s^{L*}, \lambda_0^{L*}, \lambda_1^{L*}) = (1+r)s^{L*} - c_1^{L*} \geq 0,$$

the nonnegativity conditions

$$\lambda_0^{L*} \geq 0$$

and

$$\lambda_1^{L*} \geq 0,$$

and the complementary slackness conditions

$$\lambda_0^{L*}(1 - c_0^{L*} - s^{L*}) = 0$$

and

$$\lambda_1^{L*}[(1+r)s^{L*} - c_1^{L*}] = 0.$$

c. Combine the first-order conditions to obtain

$$c_0^{L*} = \frac{1}{\lambda_1^{L*}(1+r)}$$

and

$$c_1^{L*} = \frac{\beta}{\lambda_1^{L*}}.$$

Combine the constraints to obtain

$$1 = c_0^{L*} + \frac{c_1^{L*}}{1+r}.$$

Next, substitute the first two of these last three conditions into the third to solve for

$$\lambda_1^{L*} = \frac{1+\beta}{1+r}.$$

Finally, substitute this result into the previous ones to obtain the desired solutions

$$c_0^{L*} = \frac{1}{1+\beta}$$

and

$$c_1^{L*} = \frac{\beta(1+r)}{1+\beta},$$

and substitute either of these two results into either of the constraints to obtain the desired solution

$$s^{L*} = \frac{\beta}{1+\beta}.$$

2. Optimal Borrowing

The borrower's dynamic optimization problem can be stated as

$$\max_{c_0^B, c_1^B, s^B} \ln(c_0^B) + \beta \ln(c_1^B) \text{ subject to } 0 \geq c_0^B + s^B \text{ and } 1 + (1+r)s^B \geq c_1^B.$$

a. The Lagrangian for this consumer's problem can be defined as

$$L(c_0^B, c_1^B, s^B, \lambda_0^B, \lambda_1^B) = \ln(c_0^B) + \beta \ln(c_1^B) - \lambda_0^B(c_0^B + s^B) + \lambda_1^B[1 + (1+r)s^B - c_1^B].$$

b. According to the Kuhn-Tucker theorem, the borrower's optimal choices of c_0^{B*}, c_1^{B*}, and s^{B*}, together with the associated values of λ_0^{B*} and λ_1^{B*}, must satisfy the first-order conditions

$$L_1(c_0^{B*}, c_1^{B*}, s^{B*}, \lambda_0^{B*}, \lambda_1^{B*}) = 1/c_0^{B*} - \lambda_0^{B*} = 0,$$
$$L_2(c_0^{B*}, c_1^{B*}, s^{B*}, \lambda_0^{B*}, \lambda_1^{B*}) = \beta/c_1^{B*} - \lambda_1^{B*} = 0,$$

and

$$L_3(c_0^{B*}, c_1^{B*}, s^{B*}, \lambda_0^{B*}, \lambda_1^{B*}) = -\lambda_0^{B*} + \lambda_1^{B*}(1+r) = 0$$

the constraints

$$L_4(c_0^{B*}, c_1^{B*}, s^{B*}, \lambda_0^{B*}, \lambda_1^{B*}) = -(c_0^{B*} + s^{B*}) \geq 0$$

and

$$L_5(c_0^{B*}, c_1^{B*}, s^{B*}, \lambda_0^{B*}, \lambda_1^{B*}) = 1 + (1+r)s^{B*} - c_1^{B*} \geq 0,$$

the nonnegativity conditions

$$\lambda_0^{B*} \geq 0$$

and

$$\lambda_1^{B*} \geq 0,$$

and the complementary slackness conditions

$$\lambda_0^{B*}(c_0^{B*} + s^{B*}) = 0$$

and

$$\lambda_1^{B*}[1 + (1+r)s^{B*} - c_1^{B*}] = 0.$$

c. Combine the first-order conditions to obtain

$$c_0^{B*} = \frac{1}{\lambda_1^{B*}(1+r)}$$

and

$$c_1^{B*} = \frac{\beta}{\lambda_1^{B*}}.$$

Combine the constraints to obtain

$$\frac{1}{1+r} = c_0^{B*} + \frac{c_1^{B*}}{1+r}.$$

Next, substitute the first two of these last three conditions into the third to solve for

$$\lambda_1^{B*} = 1 + \beta.$$

Finally, substitute this result into the previous ones to obtain the desired solutions

$$c_0^{B*} = \frac{1}{(1+r)(1+\beta)}$$

and

$$c_1^{B*} = \frac{\beta}{1+\beta},$$

and substitute either of these two results into either of the constraints to obtain the desired solution

$$s^{B*} = -\frac{1}{(1+r)(1+\beta)}.$$

3. Equilibrium Allocations

Equilibrium in the market for lending and borrowing requires that

$$ns^{L*} + ns^{B*} = 0.$$

a. Substituting the solutions for s^{L*} and s^{B*} obtained above into this market-clearing condition yields

$$\frac{\beta}{1+\beta} - \frac{1}{(1+r)(1+\beta)} = 0,$$

which can be solved for

$$r = 1/\beta - 1.$$

This solution indicates that when consumer's become more patient, so that β rises, the equilibrium interest rate falls.

b. Substituting this solution for the equilibrium interest rate back into the solutions for c_0^{L*}, c_1^{L*}, c_0^{B*}, c_1^{B*} yields

$$c_0^{L*} = c_1^{L*} = \frac{1}{1+\beta}$$

and

$$c_0^{B*} = c_1^{B*} = \frac{\beta}{1+\beta}$$

Evidently, both lenders and borrowers optimally choose a constant level of consumption over time.

d. Since $0 < \beta < 1$, lenders consume more during both periods. This is because their endowment of one unit of the consumption good at $t = 0$ is worth more, in present value at $t = 0$, than the borrowers' endowment of one unit of the good at $t = 1$. In other words: since lenders are wealthier, they get to consume more.

4. Optimal Allocations

Assuming that the social planner treats all agents of a given type the same during each period, let c_0^L and c_0^B denote the amount of consumption that planner gives to each lender and borrower during period $t = 0$ and, similarly, let c_1^L and c_1^B denote the amount of consumption that planner gives to each lender and borrower during period $t = 1$. The fact that there are n agents of each type and the fact that only lenders get an endowment when young and only borrowers get an endowment when old means that the planner faces the aggregate resource constraints

$$n \geq nc_0^L + nc_0^B$$

during period $t = 0$ and

$$n \geq nc_1^L + nc_1^B$$

during period $t = 1$. The planner chooses c_0^L, c_0^B, c_1^L , and c_1^B to maximize a weighted average of utilities enjoyed by a representative lender and a representative borrower:

$$\omega[\ln(c_0^L) + \beta \ln(c_1^L)] + (1-\omega)[\ln(c_0^B) + \beta \ln(c_1^B)]$$

subject to the two resource constraints, where the weight ω assigned to the lender's utility satisfies $0 < \omega < 1$.

a. After simplifying the statements of the constraints by dividing through by n as suggested, the Lagrangian for the planner's problem can be defined as

$$\begin{aligned} L(c_0^L, c_0^B, c_1^L, c_1^B, \lambda_0^P, \lambda_1^P) &= \omega[\ln(c_0^L) + \beta \ln(c_1^L)] + (1-\omega)[\ln(c_0^B) + \beta \ln(c_1^B)] \\ &\quad + \lambda_0^P(1 - c_0^L - c_0^B) + \lambda_1^P(1 - c_1^L - c_1^B). \end{aligned}$$

b. According to the Kuhn-Tucker theorem, planners's optimal choices of c_0^{L*}, c_0^{B*}, c_1^{L*}, and c_1^{B*}, together with the associated values of λ_0^{P*} and λ_1^{P*}, must satisfy the first-order conditions

$$L_1(c_0^{L*}, c_0^{B*}, c_1^{L*}, c_1^{B*}, \lambda_0^{P*}, \lambda_1^{P*}) = \omega/c_0^{L*} - \lambda_0^{P*} = 0,$$

$$L_2(c_0^{L*}, c_0^{B*}, c_1^{L*}, c_1^{B*}, \lambda_0^{P*}, \lambda_1^{P*}) = (1-\omega)/c_0^{B*} - \lambda_0^{P*} = 0,$$

$$L_3(c_0^{L*}, c_0^{B*}, c_1^{L*}, c_1^{B*}, \lambda_0^{P*}, \lambda_1^{P*}) = \omega/c_1^{L*} - \lambda_1^{P*} = 0,$$

and

$$L_4(c_0^{L*}, c_0^{B*}, c_1^{L*}, c_1^{B*}, \lambda_0^{P*}, \lambda_1^{P*}) = (1-\omega)/c_1^{B*} - \lambda_1^{P*} = 0$$

the constraints

$$L_5(c_0^{L*}, c_0^{B*}, c_1^{L*}, c_1^{B*}, \lambda_0^{P*}, \lambda_1^{P*}) = 1 - c_0^{L*} - c_0^{B*} \geq 0$$

and

$$L_6(c_0^{L*}, c_0^{B*}, c_1^{L*}, c_1^{B*}, \lambda_0^{P*}, \lambda_1^{P*}) = 1 - c_1^{L*} - c_1^{B*} \geq 0,$$

the nonnegativity conditions

$$\lambda_0^{P*} \geq 0$$

and

$$\lambda_1^{P*} \geq 0,$$

and the complementary slackness conditions

$$\lambda_0^{P*}(1 - c_0^{L*} - c_0^{B*}) = 0$$

and

$$\lambda_1^{P*}(1 - c_1^{L*} - c_1^{B*}) = 0.$$

c. The binding resource constraint for period $t = 0$ implies that $c_0^{B*} = 1 - c_0^{L*}$. Combining this condition with the first-order conditions for c_0^{B*} and c_0^{L*} yields

$$\frac{\omega}{c_0^{L*}} = \frac{1-\omega}{1 - c_0^{L*}}$$

which can be solved for

$$c_0^{L*} = \omega.$$

It then follows from the resource constraint that

$$c_0^{B*} = 1 - \omega.$$

Manipulating the period $t = 1$ conditions in the same way yields

$$c_1^{L*} = \omega.$$

It then follows from the resource constraint that

$$c_1^{B*} = 1 - \omega.$$

d. These solutions reveal that, regardless of the value of ω, optimal allocations assign a constant level of consumption to each individual agent.

e. And setting $\omega = 1/(1+\beta)$ makes the optimal allocations coincide with the equilibrium allocations solved for previously.