Solutions to Problem Set 5

ECON 772001 - Math for Economists — Peter Ireland
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Due Tuesday, October 3

1. Optimal Allocations

Consider an economy in which output is produced with capital k and labor (hours worked) h according to the Cobb-Douglas specification $k^\alpha h^{1-\alpha}$, where $0 < \alpha < 1$. In this static model, the capital stock k is taken as given, but hours worked h and consumption c are chosen by a benevolent social planner in order to maximize the utility $\ln(c) - h$ of a representative consumer, where ln denotes the natural logarithm. Hence, an optimal resource allocation solves the problem

$$\max_{h,c} \ln(c) - h \text{ subject to } k^\alpha h^{1-\alpha} \geq c.$$

Define the Lagrangian for this problem as

$$L(h, c, \lambda) = \ln(c) - h + \lambda(k^\alpha h^{1-\alpha} - c)$$

and note that the first-order conditions

$$L_1(h^*, c^*, \lambda^*) = -1 + \lambda^*(1-\alpha)k^\alpha (h^*)^{-\alpha} = 0$$

and

$$L_2(h^*, c^*, \lambda^*) = 1/c^* - \lambda^* = 0$$

can be combined with the binding constraint

$$L_3(h^*, c^*, \lambda^*) = k^\alpha (h^*)^{1-\alpha} - c^* - 0$$

to form a system of three equations in the three unknowns h^*, c^* and λ^*. Combine the two first-order conditions to obtain

$$c^* = (1-\alpha)k^\alpha (h^*)^{-\alpha}$$

and then combine this last expression with the binding constraint to obtain

$$(1-\alpha)k^\alpha (h^*)^{-\alpha} = k^\alpha (h^*)^{1-\alpha}$$

which can be used to find

$$h^* = 1 - \alpha.$$

Finally, substitute this last expression back into the binding constraint to find

$$c^* = k^\alpha (1-\alpha)^{1-\alpha}.$$

2. Equilibrium Allocations

Next, consider the same economy, but where perfectly competitive markets for inputs and outputs replace the social planner in allocating resources.

a. Now, the representative consumer (standing in for a large number of identical consumers) is endowed with k^s units of capital, and chooses labor supply h^s and consumption c to maximize utility subject to a budget constraint, that is, to solve

$$\max_{h^s,c} \ln(c) - h^s \text{ subject to } rk^s + wh^s \geq c,$$

where r is the rental rate for capital, w is the wage rate for labor, and output is the economy's numeraire, so that the price of consumption equals one and all other prices are expressed in real terms. Define the Lagrangian for this problem as

$$L(h^s, c, \lambda) = \ln(c) - h^s + \lambda(rk^s + wh^s - c)$$

and note that the first-order conditions

$$L_1(h^{s*}, c^*, \lambda^*) = -1 + \lambda^* w = 0$$

and

$$L_2(h^{s*}, c^*, \lambda^*) = 1/c^* - \lambda^* = 0$$

can be combined with the binding constraint

$$L_3(h^{s*}, c^*, \lambda^*) = rk^s + wh^{s*} - c^* = 0$$

to form a system of three equations in the three unknowns h^{s*}, c^*, and λ^*. Combine the two first-order conditions to obtain

$$c^* = w.$$

The substitution this solution into the binding constraint to find

$$h^{s*} = 1 - (r/w)k^s.$$

b. Meanwhile, a representative firm (standing in for a large number of identical firms) rents k^d units of capital and hires h^d units of labor to produce output according to the production function $(k^d)^\alpha (h^d)^{1-\alpha}$; its profit-maximization problem is

$$\max_{k^d,h^d} (k^d)^\alpha (h^d)^{1-\alpha} - rk^d - wh^d.$$

Since this is an unconstrained optimization problem, necessary conditions for a solution are

$$\alpha(k^{d*})^{\alpha-1}(h^{d*})^{1-\alpha} - r = 0$$

and

$$(1-\alpha)(k^{d*})^\alpha (h^{d*})^{-\alpha} - w = 0.$$

c. In a competitive equilibrium, market clearing for inputs and outputs requires that $k^s = k^{d*} = k$, $h^{s*} = h^{d*} = h^*$, and $c^* = k^\alpha (h^*)^{1-\alpha}$. The firm's optimality conditions therefore imply that

$$r/w = [\alpha/(1-\alpha)](h^*/k).$$

Substituting this last expression into the consumer's solution for h^{s*} yields

$$h^* = 1 - \alpha.$$

Substituting this solution for h^* back into the firm's optimality condition yields

$$w = k^\alpha (1-\alpha)^{1-\alpha}$$

so that

$$c^* = k^\alpha (1-\alpha)^{1-\alpha}.$$

d. In this case, equilibrium allocations coincide with optimal allocations: the two welfare theorems of economics apply.

3. Optimal Pollution

Now suppose that the production of $y = k^\alpha h^{1-\alpha}$ units of output yields the same amount of pollution d, and that more pollution impacts negatively on the representative consumer, whose preferences are now described by the utility function $\ln(c) - h - \gamma \ln(d)$, where $0 < \gamma < 1$. A social planner will take into account the fact that more consumption can only be obtained with more pollution, that is, that $c = d$, and will therefore solve

$$\max_{h,c} (1-\gamma)\ln(c) - h \text{ subject to } k^\alpha h^{1-\alpha} \geq c.$$

Algebraic manipulations that parallel those from question 1 above lead to the solutions

$$h^* = (1-\gamma)(1-\alpha)$$

and

$$c^* = k^\alpha [(1-\gamma)(1-\alpha)]^{1-\alpha}.$$

Since $0 < \gamma < 1$, output, employment, and consumption are lower than before, since the social planner optimally accounts for the negative effects of pollution.

4. Negative Externalities

Now assume once again that allocations are determined by perfectly competitive markets, but that individual consumers fail to take into account the fact that the more they choose to consume the more pollution there will be and that, similarly, individual firms are not penalized in any way for the pollution that they create.

a. Now, consistent with the idea that no one individual views him or herself as being able to do anything about the total amount of pollution economy-wide, the representative consumer solves

$$\max_{h^s,c} \ln(c) - h^s - \gamma \ln(d) \text{ subject to } rk^s + wh^s \geq c,$$

taking d as a given. The solutions

$$c^* = w$$

and

$$h^{s*} = 1 - (r/w)k^s.$$

are the same as before, but the consumer enjoys lower levels of utility because of the economy-wide level of pollution.

b. Also consistent with the idea that individual businesses are not penalized for the pollution they create, the representative firm solves

$$\max_{k^d, h^d} (k^d)^\alpha (h^d)^{1-\alpha} - rk^d - wh^d,$$

exactly as before. Its optimal decisions are described by the the conditions

$$\alpha(k^{d*})^{\alpha-1}(h^{d*})^{1-\alpha} - r = 0$$

and

$$(1-\alpha)(k^{d*})^\alpha (h^{d*})^{-\alpha} - w = 0.$$

just as before.

c. Again as before, equilibrium allocations are described by

$$h^* = 1 - \alpha.$$

and

$$c^* = k^\alpha (1-\alpha)^{1-\alpha}.$$

d. Since no individual consumer accounts for the negative externality that his or her consumption generates, and since no individual firm accounts for the negative externality that its production generates, there is too much production and pollution in the market economy.

5. Government Intervention

Finally, suppose that the government intervenes in the market economy by taxing firms at the rate τ for every unit of output – and hence every unit of pollution – they create and using the proceeds of this output tax to provide each consumer with a payment T that compensates him or her for having to suffer the ill effects of pollution.

a. Now the representative consumer takes the amount of pollution d and the government payment T as given, and solves

$$\max_{h^s, c} \ln(c) - h^s - \gamma \ln(d) \text{ subject to } T + rk^s + wh^s \geq c.$$

taking d as a given. Setting up the Lagrangian and using the first-order conditions and binding constraint as before leads to the solutions

$$c^* = w$$

and

$$h^{s*} = 1 - (T + rk^s)/w.$$

b. Meanwhile, the representative firm acts to maximize its after-tax profits by solving

$$\max_{k^d,h^d} (1-\tau)(k^d)^\alpha (h^d)^{1-\alpha} - rk^d - wh^d.$$

The optimality conditions that characterize the solution to this unconstrained optimization problem are

$$(1-\tau)\alpha(k^{d*})^{\alpha-1}(h^{d*})^{1-\alpha} - r = 0$$

and

$$(1-\tau)(1-\alpha)(k^{d*})^\alpha (h^{d*})^{-\alpha} - w = 0.$$

c. Substitute the firm's optimality conditions and the government budget constraint $T = \tau k^\alpha (h^*)^{1-\alpha}$ into the solution for h^{s*} from the consumer's problem to obtain the expression

$$h^* = 1 - \frac{\tau k^\alpha (h^*)^{1-\alpha} + (1-\tau)\alpha k^\alpha (h^*)^{1-\alpha}}{(1-\tau)(1-\alpha)k^\alpha (h^*)^{-\alpha}} = 1 - \left[\frac{\tau + (1-\tau)\alpha}{(1-\tau)(1-\alpha)}\right] h^*,$$

which can be solved for

$$h^* = (1-\tau)(1-\alpha).$$

Then combine this result and the firm's optimality conditions into the solution for c^* from the consumer's problem to obtain

$$c^* = k^\alpha [(1-\tau)(1-\alpha)]^{1-\alpha}.$$

d. Comparing the solutions just shown to the solutions to the social planner's problem reveals that the government can induce firms to internalize the negative effects that pollution has on consumers by taxing output at the rate $\tau = \gamma$.