Solutions to Problem Set 4

ECON 772001 - Math for Economists — Peter Ireland
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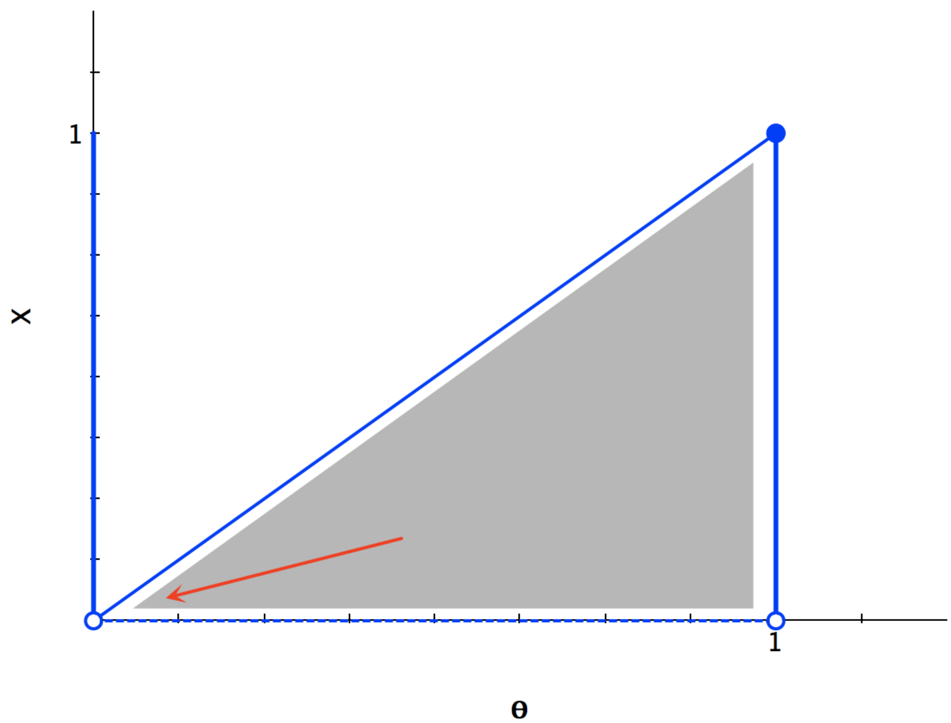
For Extra Practice – Not Collected or Graded

1. Upper Hemicontinuity

Let $\Theta = [0,1] \subseteq \mathbf{R}$ and $X = [0,1] \subseteq \mathbf{R}$, and consider the correspondence $G : \Theta \to X$ defined by

$$G(\theta) = \begin{cases} (0,1] & \text{for} \quad \theta = 0 \\ (0,\theta] & \text{for} \quad \theta \in (0,1]. \end{cases}$$

a. The blue lines in the figure below graph $G(\theta)$ with Θ on the horizontal axis and X on the vertical axis.



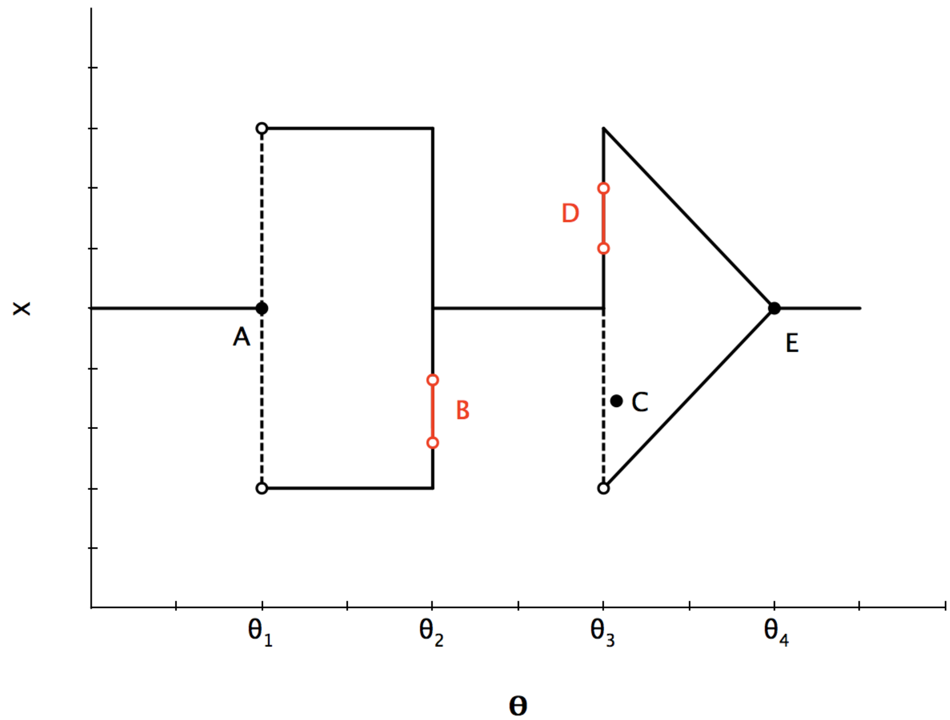
b. G upper hemicontinuous. To see this, fix any value for $\theta \in [0,1]$ and consider an open subset X' of X that contains $G(\theta)$. For any θ' that lies slightly to the left or right of θ, $G(\theta')$ will also be continued in X'.

c. Let $\{\theta_j\}$ be a sequence with $\theta_j \to 0$ and let $\{x_j\}$ be a sequence with $x_j \in G(\theta_j)$ for all j. From red line that is also shown in the figure, it is clear that any such sequence will have to converge to $x = 0$, but $x = 0$ is not an element of $G(0)$. How can we reconcile this observation with the fact, noted above, that G is upper hemicontinuous?

The answer is that G is not compact valued, so that the alternative definition of upper hemicontinuity that is cast in terms of sequences instead of sets, does not apply.

2. Upper and Lower Hemicontinuity

The graph below shows the graph of a compact-valued correspondence $G : \Theta \to X$, where $\Theta \subseteq \mathbf{R}$ and $X \subseteq \mathbf{R}$.



a. Is G upper hemicontinuous at θ_1? No! Choose a sufficiently small open set containing point A and move slightly to the right: $G(\theta)$ is no longer contained in that set. Is G lower hemicontinuous at θ_1? Yes! Chose any open set containing point A and move slightly to the left or the right: that open set will still intersect with $G(\theta')$.

b. Is G upper hemicontinuous at θ_2? Yes! Choose any open set containing $G(\theta_2)$ and move slightly to the left or the right: $G(\theta')$ will still be contained in that open set. Is G lower hemicontinuous at θ_2? No! Choose an open set like the one labelled B in the graph, then move slightly to the right: $G(\theta')$ no longer intersects with B.

c. Is G upper hemicontinuous at θ_3? No! Chose a sufficiently small open set containing $G(\theta_3)$ and move slightly to the right: there will be points like C that are contained in $G(\theta)$ but not in the open set. Is G lower hemicontinuous at θ_3? No! Choose an open set like the one labelled D in the growth and move slightly to the left: $G(\theta')$ no longer intersects with D.

d. Is G upper hemicontinuous at θ_4? Yes! Choose any open set containing point E and move slightly to the left or the right: $G(\theta')$ will still be contained in that open set. Is G lower hemicontinuous at θ_4? Yes! Choose any open set containing point E and move slightly to the left or the right: that open set will still intersect with $G(\theta')$.

3. Pareto Optimal Allocations

Given $k \in K = [\underline{k}, \bar{k}]$, with $0 < \underline{k} < \bar{k} < \infty$, the social planner's problem is

$$\max_{c_1, c_2} \omega u(c_1) + (1-\omega)v(c_2) \text{ subject to } f(k) \geq c_1 + c_2,\ c_1 \geq 0,\ c_2 \geq 0,$$

where $0 < \omega < 1$ and the utility functions u and v and the production function f are all continuous. Clearly, the correspondence

$$G(k) = \left\{(c_1, c_2) \in \mathbf{R}^2 \mid f(k) \geq c_1 + c_2,\ c_1 \geq 0,\ c_2 \geq 0\right\}$$

is nonempty and compact valued, but to apply Berge's maximum theorem, we still need to confirm that G is upper and lower hemicontinuous.

To show that G is upper hemicontinuous, fix $k \in K$, let $\{k_j\}$ be a sequence with $k_j \to k$, and let $\{c_{1j}, c_{2j}\}$ be a sequence with $(c_{1j}, c_{2j}) \in G(k_j)$ for all j.

Is there a subsequence $\{c_{1j_n}, c_{2j_n}\}$ of $\{c_{1j}, c_{2j}\}$ with $(c_{1j_n}, c_{2j_n}) \to (c_1, c_2) \in G(k)$? Yes!

To see this, observe first that since $k_j \to k$, there exists a closed and bounded subset $\hat{K}$ of K such that, for some $J \geq 1$, all of the k_j, $j \geq J$, and k are contained in $\hat{K}$. Observe, next, that given the structure of G, all of the (c_{1j}, c_{2j}) for $j \geq J$ will lie in some closed and bounded subset of $\mathbf{R}^2$. Thus, for all $j \geq J$, all elements of the sequence $\{k_j, c_{1j}, c_{2j}\}$ will lie in a closed and bounded subset of $\mathbf{R}^3$ and, by the Bolzano-Weierstrass theorem, this sequence will have a convergent subsequence $\{k_{j_n}, c_{1j_n}, c_{2j_n}\}$ with limit point (k, c_1, c_2). And since each element of this convergent subsequence satisfies

$$f(k_{j_n}) \geq c_{1j_n} + c_{2j_n},$$

$c_{1j_n} \geq 0$ and $c_{2j_n} \geq 0$, the limit point will also satisfy $(c_1, c_2) \in G(k)$.

To show that G is lower hemicontinuous as well, fix $k \in K$ and $(c_1, c_2) \in G(k)$, then let $\{k_j\}$ be a sequence such that $k_j \to k$.

Is there a sequence $\{c_{1j}, c_{2j}\}$ with $(c_{1j}, c_{2j}) \in G(k_j)$ for all j and $(c_{1j}, c_{2j}) \to (c_1, c_2)$? Yes!

To see this, suppose first that $c_1 = c_2 = 0$. In this case, the sequence with $c_{1j} = c_{2j} = 0$ for all j with work.

So suppose instead that $c_1 > 0$ and/or $c_2 > 0$. For this case, let

$$c_{1j} = \left(\frac{f(k_j)}{f(k)}\right) c_1$$

and

$$c_{2j} = \left(\frac{f(k_j)}{f(k)}\right) c_2$$

for all j. Note that $c_{1j} \geq 0$ and $c_{2j} \geq 0$ for all j. Moreoever,

$$c_{1j} + c_{2j} = \left(\frac{k_j}{f(k)}\right)(c_1 + c_2) \leq f(k_j),$$

so that $(c_{1j}, c_{2j}) \in G(k_j)$ for all j. Moreover, the continuity of f implies that $(c_{1j}, c_{2j}) \to (c_1, c_2)$, completing the proof.

We can now conclude, from Berge's maximum theorem, that the maximum value function

$$W(k) = \max_{(c_1,c_2)\in G(k)} \omega u(c_1) + (1-\omega)v(c_2)$$

is well-defined and continuous and that the optimal policy correspondence

$$c^*(k) = \{(c_1^*, c_2^*) \in G(k) \mid \omega u(c_1^*) + (1-\omega)v(c_2^*) = W(k)\}$$

is nonempty, compact-valued, and upper hemicontinuous. Finally, if we are willing to assume, as well, that u and v are both strictly concave, then $c_1^*(k)$ and $c_2^*(k)$ will both be continuous functions of k.