**Solutions to Problem Set 4**

ECON 772001 - Math for Economists — Peter Ireland
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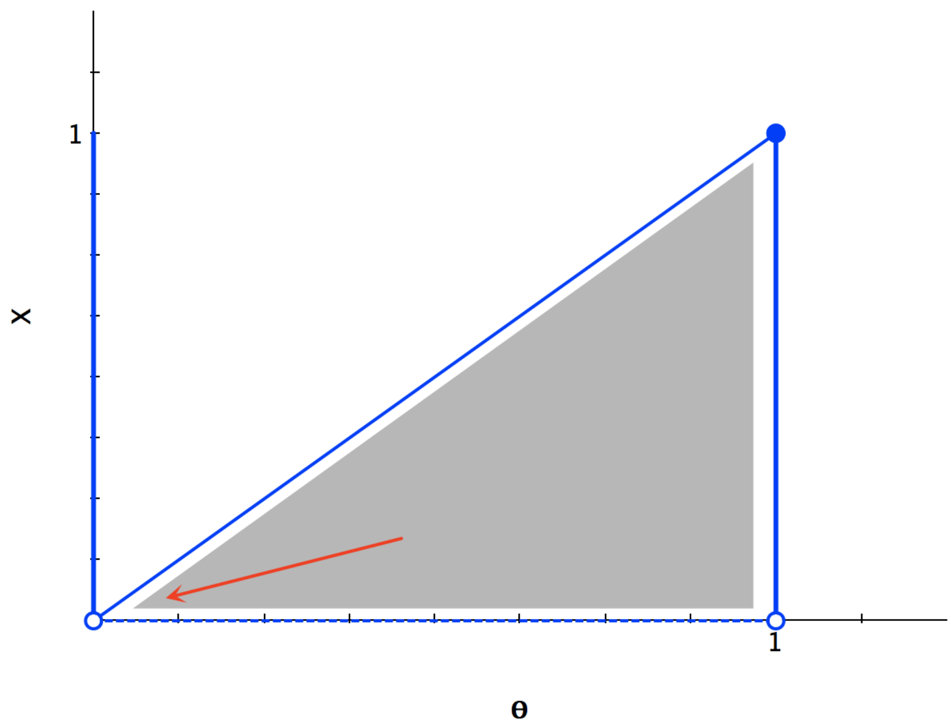
For Extra Practice – Not Collected or Graded

## 1. Upper Hemicontinuity

Let $\Theta = [0,1] \subseteq \mathbf{R}$ and $X = [0,1] \subseteq \mathbf{R}$, and consider the correspondence $G : \Theta \to X$ defined by

$$G(\theta) = \begin{cases} (0,1] & \text{for} \quad \theta = 0 \\ (0,\theta] & \text{for} \quad \theta \in (0,1]. \end{cases}$$

a. The blue lines in the figure below graph $G(\theta)$ with $\Theta$ on the horizontal axis and $X$ on the vertical axis.



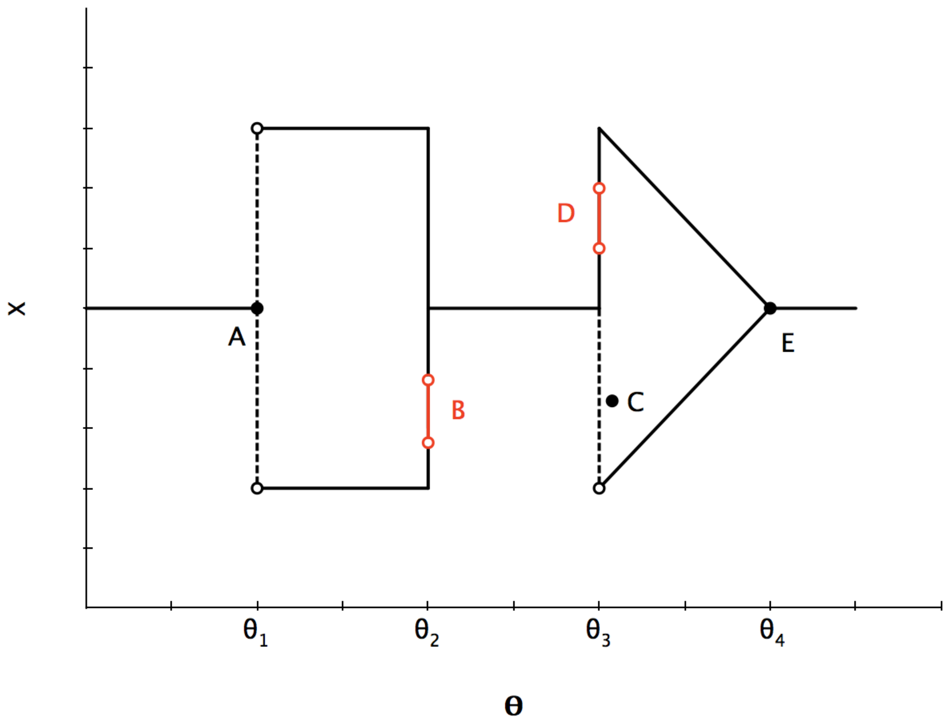

b. $G$ upper hemicontinuous. To see this, fix any value for $\theta \in [0,1]$ and consider an open subset $X'$ of $X$ that contains $G(\theta)$. For any $\theta'$ that lies slightly to the left or right of $\theta$, $G(\theta')$ will also be continued in $X'$.

c. Let $\{\theta_j\}$ be a sequence with $\theta_j \to 0$ and let $\{x_j\}$ be a sequence with $x_j \in G(\theta_j)$ for all $j$. From red line that is also shown in the figure, it is clear that any such sequence will have to converge to $x = 0$, but $x = 0$ is not an element of $G(0)$. How can we reconcile this observation with the fact, noted above, that $G$ is upper hemicontinuous?

The answer is that $G$ is not compact valued, so that the alternative definition of upper hemicontinuity that is cast in terms of sequences instead of sets does not apply.

**2. Upper and Lower Hemicontinuity**

The graph below shows the graph of a compact-valued correspondence $G : \Theta \to X$, where $\Theta \subseteq \mathbf{R}$ and $X \subseteq \mathbf{R}$.




a. Is $G$ upper hemicontinuous at $\theta_1$? No! Choose a sufficiently small open set containing point A and move slightly to the right: $G(\theta)$ is no longer contained in that set. Is $G$ lower hemicontinuous at $\theta_1$? Yes! Chose any open set containing point A and move slightly to the left or the right: that open set will still intersect with $G(\theta')$.

b. Is $G$ upper hemicontinuous at $\theta_2$? Yes! Choose any open set containing $G(\theta_2)$ and move slightly to the left or the right: $G(\theta')$ will still be contained in that open set. Is $G$ lower hemicontinuous at $\theta_2$? No! Choose an open set like the one labelled B in the graph, then move slightly to the right: $G(\theta')$ no longer intersects with B.

c. Is $G$ upper hemicontinuous at $\theta_3$? No! Chose a sufficiently small open set containing $G(\theta_3)$ and move slightly to the right: there will be points like C that are contained in $G(\theta)$ but not in the open set. Is $G$ lower hemicontinuous at $\theta_3$? No! Choose an open set like the one labelled $D$ in the growth and move slightly to the left: $G(\theta')$ no longer intersects with D.

d. Is $G$ upper hemicontinuous at $\theta_4$? Yes! Choose any open set containing point E and move slightly to the left or the right: $G(\theta')$ will still be contained in that open set. Is $G$ lower hemicontinuous at $\theta_4$? Yes! Choose any open set containing point E and move slightly to the left or the right: that open set will still intersect with $G(\theta')$.

**3. Pareto Optimal Allocations**

Given $k \in K = [\underline{k}, \bar{k}]$, with $0 < \underline{k} < \bar{k} < \infty$, the social planner's problem is

$$\max_{c_1, c_2} \omega u(c_1) + (1-\omega)v(c_2) \text{ subject to } f(k) \geq c_1 + c_2,\ c_1 \geq 0,\ c_2 \geq 0,$$

where $0 < \omega < 1$ and the utility functions $u$ and $v$ and the production function $f$ are all continuous. Clearly, the correspondence

$$G(k) = \big\{(c_1, c_2) \in \mathbf{R}^2 \mid f(k) \geq c_1 + c_2,\ c_1 \geq 0,\ c_2 \geq 0\big\}$$

is nonempty and compact valued, but to apply Berge's maximum theorem, we still need to confirm that $G$ is upper and lower hemicontinuous.

To show that $G$ is upper hemicontinuous, fix $k \in K$, let $\{k_j\}$ be a sequence with $k_j \to k$, and let $\{c_{1j}, c_{2j}\}$ be a sequence with $(c_{1j}, c_{2j}) \in G(k_j)$ for all $j$.

Is there a subsequence $\{c_{1j_n}, c_{2j_n}\}$ of $\{c_{1j}, c_{2j}\}$ with $(c_{1j_n}, c_{2j_n}) \to (c_1, c_2) \in G(k)$? Yes!

To see this, observe first that since $k_j \to k$, there exists a bounded subset $\hat{K}$ of $K$ such that, for some $J \geq 1$, all of the $k_j$, $j \geq J$, and $k$ are contained in $\hat{K}$. Observe, next, that given the structure of $G$, all of the $(c_{1j}, c_{2j})$ for $j \geq J$ will lie in some bounded subset of $\mathbf{R}^2$. Thus, for all $j \geq J$, all elements of the sequence $\{k_j, c_{1j}, c_{2j}\}$ will lie in a bounded subset of $\mathbf{R}^3$ and, by the Bolzano-Weierstrass theorem, this sequence will have a convergent subsequence $\{k_{j_n}, c_{1j_n}, c_{2j_n}\}$ with limit point $(k, c_1, c_2)$. And since each element of this convergent subsequence satisfies

$$f(k_{j_n}) \geq c_{1j_n} + c_{2j_n},$$

$c_{1j_n} \geq 0$ and $c_{2j_n} \geq 0$, the limit point will also satisfy $(c_1, c_2) \in G(k)$.

To show that $G$ is lower hemicontinuous as well, fix $k \in K$ and $(c_1, c_2) \in G(k)$, then let $\{k_j\}$ be a sequence such that $k_j \to k$.

Is there a sequence $\{c_{1j}, c_{2j}\}$ with $(c_{1j}, c_{2j}) \in G(k_j)$ for all $j \geq J$ for some $J \geq 1$ and $(c_{1j}, c_{2j}) \to (c_1, c_2)$? Yes!

To see this, let

$$c_{1j} = \left[\frac{f(k_j)}{f(k)}\right] c_1$$

and

$$c_{2j} = \left[\frac{f(k_j)}{f(k)}\right] c_2$$

for all $j$. Since $k_j \to k \in K$ and since $f : K \to (0, \infty)$ is continuous, there exists some $J \geq 1$ so that $f(k_j) > 0$ and hence $c_{1j} \geq 0$ and $c_{2j} \geq 0$ for all $j \geq J$. Moreoever,

$$c_{1j} + c_{2j} = \left[\frac{f(k_j)}{f(k)}\right](c_1 + c_2) \leq f(k_j),$$

so that $(c_{1j}, c_{2j}) \in G(k_j)$, for all $j \geq J$. Finally, the continuity of $f$ implies that $(c_{1j}, c_{2j}) \to (c_1, c_2)$, completing the proof.

We can now conclude, from Berge's maximum theorem, that the maximum value function

$$W(k) = \max_{(c_1, c_2) \in G(k)} \omega u(c_1) + (1 - \omega) v(c_2)$$

is well-defined and continuous and that the optimal policy correspondence

$$c^*(k) = \{(c_1^*, c_2^*) \in G(k) \mid \omega u(c_1^*) + (1 - \omega) v(c_2^*) = W(k)\}$$

is nonempty, compact-valued, and upper hemicontinuous. Finally, if we are willing to assume, as well, that $u$ and $v$ are both strictly concave, then $c_1^*(k)$ and $c_2^*(k)$ will both be continuous functions of $k$.