Solutions to Problem Set 3

ECON 772001 - Math for Economists — Peter Ireland
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1. Hotelling's Lemma

Consider a firm that produces output y with capital k and labor l according to the technology described by

$$f(k, l) \geq y.$$

The firm sells each unit of output at the price p, rents each unit of capital at the rate r, and hires each unit of labor at the wage w. Hence it chooses y, k, and l to maximize profits

$$py - rk - wl$$

subject to the technological constraint just shown above.

a. The Lagrangian for the firm's problem is

$$L(y, k, l, \lambda) = py - rk - wk + \lambda[f(k, l) - y].$$

b. According to the Kuhn-Tucker theorem, the values y^*, k^*, and l^* that solve the firm's problem, together with the associated value λ^* for the multiplier, must satisfy the first-order conditions

$$L_1(y^*, k^*, l^*, \lambda^*) = p - \lambda^* = 0,$$

$$L_2(y^*, k^*, l^*, \lambda^*) = -r + \lambda^* f_1(k^*, l^*) = 0,$$

and

$$L_3(y^*, k^*, l^*, \lambda^*) = -w + \lambda^* f_2(k^*, l^*) = 0,$$

the constraint

$$L_4(y^*, k^*, l^*, \lambda^*) = f(k^*, l^*) - y^* \geq 0,$$

the nonnegativity condition

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^*[f(k^*, l^*) - y^*] = 0.$$

c. Assume that the output and input prices p, r, and w and the production function f are such that it is possible to solve uniquely for the values of y^*, k^*, l^*, and λ^* in terms of the parameters p, r, and w. Then the profit function, defined as

$$\pi(p, r, w) = \max_{y,k,l} py - rk - wl \text{ subject to } f(k, l) \geq y,$$

can be evaluated as

$$\pi(p,r,w) = py^*(p,r,w) - rk^*(p,r,w) - wl^*(p,r,w)$$

or, using the complementary slackness condition,

$$\begin{aligned}\pi(p,r,w) &= py^*(p,r,w) - rk^*(p,r,w) - wl^*(p,r,w) \\ &\quad + \lambda^*(p,r,w)\{f[k^*(p,r,w), l^*(p,r,w)] - y^*(p,r,w)\},\end{aligned}$$

where $y^*(p,r,w)$ is the firm's supply function, $k^*(p,r,w)$ and $l^*(p,r,w)$ are the factor demand curves, and $\lambda^*(p,r,w)$ is the function describing the associated values of the Lagrange multiplier. The envelope theorem says that in differentiating this expression for the profit function through by each argument, one can ignore the dependence of y^*, k^*, l^*, and λ^* on those parameters, and simply write

$$\pi_1(p,r,w) = y^*(p,r,w),$$

$$\pi_2(p,r,w) = -k^*(p,r,w),$$

and

$$\pi_3(p,r,w) = -l^*(p,r,w),$$

which are the same results described by Hotelling's lemma.

2. Shephard's Lemma

Closely related to the profit maximization problem from above is the corresponding cost minimization problem in which the same firm chooses capital k and labor l inputs to minimize the cost of producing y units of output:

$$\min_{k,l} rk + wl \text{ subject to } f(k,l) \geq y.$$

a. The Lagrangian for this problem is

$$L(k,l,\mu) = rk + wl - \mu[f(k,l) - y].$$

b. According to the Kuhn-Tucker theorem, the values k^* and l^* that solve the firm's problem, together with the associated value μ^* for the multiplier, must satisfy the first-order conditions

$$L_1(k^*,l^*,\mu^*) = r - \mu^* f_1(k^*,l^*) = 0$$

and

$$L_2(k^*,l^*,\mu^*) = w - \mu^* f_2(k^*,l^*) = 0,$$

the constraint

$$L_3(k^*,l^*,\mu^*) = y - f(k^*,l^*) \leq 0,$$

the nonnegativity condition

$$\mu^* \geq 0,$$

and the complementary slackness condition

$$\mu^*[f(k^*,l^*) - y] = 0.$$

c. Assume that the output requirement y, the input prices r and w, and the production function f are such that it is possible to solve uniquely for the values of k^*, l^*, and μ^* in terms of the parameters r, w, and y. Then the cost function, defined as

$$c(r, w, y) = \min_{k,l} rk + wl \text{ subject to } f(k, l) \geq y,$$

can be evaluated as

$$c(r, w, y) = rk^*(r, w, y) + wl^*(r, w, y)$$

or, using the complementary slackness condition,

$$c(r, w, y) = rk^*(r, w, y) + wl^*(r, w, y) - \mu^*(r, w, y)\{f[k^*(r, w, y), l^*(r, w, y)] - y\},$$

where $k^*(r, w, y)$ and $l^*(r, w, y)$ now represent the conditional factor demand curves and $\mu^*(r, w, y)$ is the function describing the associated values of the Lagrange multiplier. The envelope theorem says that in differentiating this expression for the cost function through by each argument, one can ignore the dependence of k^*, l^*, and μ^* on those parameters, and simply write

$$c_1(r, w, y) = k^*(r, w, y)$$

and

$$c_2(r, w, y) = l^*(r, w, y),$$

which are the same results described by Shephard's lemma.

3. Roy's Identity

Now consider a utility-maximizing consumer who solves

$$\max_{c_1,c_2} U(c_1, c_2) \text{ subject to } I \geq p_1c_1 + p_2c_2,$$

where c_1 and c_2 denote consumption of two goods, U is the utility function, I is income, and p_1 and p_2 are the prices of the two goods.

a. The Lagrangian for this problem is

$$L(c_1, c_2, \lambda) = U(c_1, c_2) + \lambda(I - p_1c_1 - p_2c_2).$$

b. According to the Kuhn-Tucker theorem implies that, the values c_1^* and c_2^* that solve the consumer's problem, together with the associated value λ^* for the multiplier, must satisfy the first-order conditions

$$L_1(c_1^*, c_2^*, \lambda^*) = U_1(c_1^*, c_2^*) - \lambda^*p_1 = 0$$

and

$$L_2(c_1^*, c_2^*, \lambda^*) = U_2(c_1^*, c_2^*) - \lambda^*p_2 = 0,$$

the constraint

$$L_3(c_1^*, c_2^*, \lambda^*) = I - p_1 c_1^* - p_2 c_2^* \geq 0,$$

the nonnegativity condition

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^*(I - p_1 c_1^* - p_2 c_2^*) = 0.$$

c. Assume that income I, and goods prices p_1 and p_2, and the utility function U are such that it is possible to solve uniquely for the values of c_1^*, c_2^*, and λ^* in terms of the parameters I, p_1, and p_2. Then the indirect utility function, defined as

$$v(p_1, p_2, I) = \max_{c_1, c_2} U(c_1, c_2) \text{ subject to } I \geq p_1 c_1 + p_2 c_2,$$

can be evaluated as

$$v(p_1, p_2, I) = U[c_1^*(p_1, p_2, I), c_2^*(p_1, p_2, I)]$$

or, using the complementary slackness condition,

$$\begin{aligned} v(p_1, p_2, I) \quad = \quad & U[c_1^*(p_1, p_2, I), c_2^*(p_1, p_2, I)] \\ & + \lambda^*(p_1, p_2, I)[I - p_1 c_1^*(p_1, p_2, I) - p_1 c_2^*(p_1, p_2, I)], \end{aligned}$$

where $c_1^*(p_1, p_2, I)$ and $c_2^*(p_1, p_2, I)$ define the Marshallian demand curves for the two goods and $\lambda^*(p_1, p_2, I)$ describes the associated values for the Lagrange multiplier. The envelope theorem says that in differentiating this last expression for the indirect utility function through by each argument, one can ignore the dependence of c_1^*, c_2^*, and λ^* on those parameters, and simply write

$$v_1(p_1, p_2, I) = -\lambda^*(p_1, p_2, I) c_1^*(p_1, p_2, I),$$

$$v_2(p_1, p_2, I) = -\lambda^*(p_1, p_2, I) c_2^*(p_1, p_2, I),$$

and

$$v_3(p_1, p_2, I) = \lambda^*(p_1, p_2, I).$$

Dividing the first and second of these equations by the third leads directly to a statement of Roy's identity.

4. The Slutsky Equation

Closely related to the utility maximization problem from above is the corresponding expenditure minimization problem in which the same consumer chooses consumptions c_1 and c_2 to minimize the cost of achieving a given utility level $\bar{U}$:

$$\min_{c_1, c_2} p_1 c_1 + p_2 c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U}.$$

a. The Lagrangian for this problem is

$$L(c_1, c_2, \mu) = p_1 c_1 + p_2 c_2 - \mu[U(c_1, c_2) - \bar{U}].$$

b. According to the Kuhn-Tucker theorem, the values c_1^* and c_2^* that solve this problem, together with the associated value μ^* for the multiplier, must satisfy the first-order conditions

$$L_1(c_1^*, c_2^*, \mu^*) = p_1 - \mu^* U_1(c_1^*, c_2^*) = 0$$

and

$$L_2(c_1^*, c_2^*, \mu^*) = p_2 - \mu^* U_2(c_1^*, c_2^*) = 0,$$

the constraint

$$L_3(c_1^*, c_2^*, \mu^*) = \bar{U} - U(c_1^*, c_2^*) \leq 0,$$

the nonnegativity condition

$$\mu^* \geq 0,$$

and the complementary slackness condition

$$\mu^*[U(c_1^*, c_2^*) - \bar{U}] = 0.$$

c. Assume that the utility level $\bar{U}$, goods prices p_1 and p_2, and the utility function U are such that it is possible to solve uniquely for the values of c_1^*, c_2^*, and μ^* in terms of the parameters $\bar{U}$, p_1, and p_2. Now the functions $c_1^* = h_1^*(p_1, p_2, \bar{U})$ and $c_2^* = h_2^*(p_1, p_2, \bar{U})$ describing the optimal choices are the Hicksian demand curves for the two goods. Along with these functions, define the expenditure function

$$e(p_1, p_2, \bar{U}) = \min_{c_1, c_2} p_1 c_1 + p_2 c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U}.$$

Under most circumstances, it will be the case that the Marshallian and Hicksian demand curves coincide at the point where $I = e(p_1, p_2, \bar{U})$, so that the income I from the utility maximization problem equals the expenditure required to attain the utility level $v(p_1, p_2, I) = \bar{U}$. This result can be summarized by stating that

$$h_i^*(p_1, p_2, \bar{U}) = c_i^*(p_1, p_2, e(p_1, p_2, \bar{U}))$$

for all values of p_1, p_2, and $\bar{U}$ and each $i = 1, 2$. Differentiating both sides of this expression by p_i and using the chain rule on the right-hand side, yields

$$\frac{\partial h_i^*(p_1, p_2, \bar{U})}{\partial p_i} = \frac{\partial c_i^*(p_1, p_2, e(p_1, p_2, \bar{U}))}{\partial p_i} + \frac{\partial c_i^*(p_1, p_2, e(p_1, p_2, \bar{U}))}{\partial I} \frac{\partial e(p_1, p_2, \bar{U})}{\partial p_i}.$$

However, the envelope theorem, when applied to evaluate the derivatives of the expenditure function, implies that

$$\frac{\partial e(p_1, p_2, \bar{U})}{\partial p_i} = h_i^*(p_1, p_2, \bar{U}) = c_i^*(p_1, p_2, e(p_1, p_2, \bar{U})).$$

Substituting this last expression, together with $I = e(p_1, p_2, \bar{U})$ and $v(p_1, p_2, I) = \bar{U}$, into the one above it and rearranging yields the Slutsky equation

$$\frac{\partial c_i^*(p_1, p_2, I)}{\partial p_i} = \frac{\partial h_i^*(p_1, p_2, v(p_1, p_2, I))}{\partial p_i} - \frac{\partial c_i^*(p_1, p_2, I)}{\partial I} c_i^*(p_1, p_2, I)$$

for each $i = 1, 2$.