Solutions to Problem Set 2

ECON 772001 - Math for Economists — Peter Ireland
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1. Utility Maximization - Second-Order Conditions

The following result specializes Theorem 19.8 from Simon and Blume's book to provide first and second-order conditions for a constrained optimization problem with two choice variables and a single constraint that is assumed to bind at the optimum.

Theorem Let $F : \mathbf{R}^2 \rightarrow \mathbf{R}$ and $G : \mathbf{R}^2 \rightarrow \mathbf{R}$ be twice continuously differentiable functions, and consider the constrained optimization problem

$$\max_{x_1,x_2} F(x_1, x_2) \text{ subject to } c \geq G(x_1, x_2),$$

with parameter $c \in \mathbf{R}$. Associated with this problem, define the Lagrangian

$$L(x_1, x_2, \lambda) = F(x_1, x_2) + \lambda[c - G(x_1, x_2)].$$

Suppose there exist values x_1^*, x_2^*, and λ^* of x_1, x_2, and λ that satisfy the first-order conditions

$$L_1(x_1^*, x_2^*, \lambda^*) = F_1(x_1^*, x_2^*) - \lambda^* G_1(x_1^*, x_2^*) = 0,$$

$$L_2(x_1^*, x_2^*, \lambda^*) = F_2(x_1^*, x_2^*) - \lambda^* G_2(x_1^*, x_2^*) = 0,$$

$$L_3(x_1^*, x_2^*, \lambda^*) = c - G(x_1^*, x_2^*) \geq 0,$$

$$\lambda^* \geq 0,$$

and

$$\lambda^*[c - G(x_1^*, x_2^*)] = 0.$$

Suppose also that $c - G(x_1^*, x_2^*)$, so that the constraint binds at the optimum, and that the "bordered Hessian" matrix

$$H = \begin{bmatrix} 0 & G_1(x_1^*, x_2^*) & G_2(x_1^*, x_2^*) \\ G_1(x_1^*, x_2^*) & L_{11}(x_1^*, x_2^*, \lambda^*) & L_{21}(x_1^*, x_2^*, \lambda^*) \\ G_2(x_1^*, x_2^*) & L_{12}(x_1^*, x_2^*, \lambda^*) & L_{22}(x_1^*, x_2^*, \lambda^*) \end{bmatrix}$$

satisfies the second-order condition that $|H| > 0$, so that the determinant of H is strictly positive. Then x_1^* and x_2^* are local maximizers of $F(x_1, x_2)$ subject to $c \geq G(x_1, x_2)$.

Note that this result provides sufficient conditions for a solution to the problem: it says that if the first and second-order conditions are satisfied, then the values of x_1^* and x_2^* constitute at least a local maximum.

With this result in mind, return to the problem solved by a consumer who uses his or her income I to purchase c_1 units of good 1 at the price of p_1 per unit and c_2 units of good 2 at the price of p_2 per unit to maximize utility

$$U(c_1, c_2) = a\ln(c_1) + (1-a)\ln(c_2),$$

with $0 < a < 1$, subject to the budget constraint

$$I \geq p_1c_1 + p_2c_2.$$

a. The Lagrangian for the consumer's problem is

$$L(c_1, c_2, \lambda) = a\ln(c_1) + (1-a)\ln(c_2) + \lambda(I - p_1c_1 - p_2c_2).$$

b. From question 3 of problem set 1, we know that

$$c_1^* = \frac{aI}{p_1},$$

$$c_2^* = \frac{(1-a)I}{p_2},$$

and

$$\lambda^* = \frac{1}{I}.$$

c. Following the pattern shown in the theorem, the bordered Hessian matrix for this problem is

$$H = \begin{bmatrix} 0 & p_1 & p_2 \\ p_1 & -\frac{a}{(c_1^*)^2} & 0 \\ p_2 & 0 & -\frac{1-a}{(c_2^*)^2} \end{bmatrix} = \begin{bmatrix} 0 & p_1 & p_2 \\ p_1 & -\frac{p_1^2}{aI^2} & 0 \\ p_2 & 0 & -\frac{p_2^2}{(1-a)I^2} \end{bmatrix}$$

and therefore satisfies the second-order condition

$$|H| = \frac{(1-a)p_1^2}{(c_2^*)^2} + \frac{ap_2^2}{(c_1^*)^2} = \frac{p_1^2p_2^2}{(1-a)I^2} + \frac{p_1^2p_2^2}{aI^2} = \frac{2p_1^2p_2^2}{a(1-a)I^2} > 0$$

confirming that the values of c_1^* and c_2^* do, in fact, solve the constrained maximization problem.

2. Complementary Slackness

Consider the constrained optimization problem

$$\max_x -x^2 \text{ subject to } x \geq 0,$$

where $x \in \mathbf{R}$ is a single choice variable (note that there is a minus sign out in front of the objective function).

a. Since the objective function is negative for any other choice of x, the solution is $x^* = 0$.

b. The Lagrangian for the problem is

$$L(x, \lambda) = -x^2 + \lambda x.$$

c. According to the Kuhn-Tucker theorem, the value x^* of x that solves the problem together with the associated value λ^* of λ must satisfy the first-order condition

$$L_1(x^*, \lambda^*) = -2x^* + \lambda^* = 0,$$

the constraint

$$L_2(x^*, \lambda^*) = x^* \geq 0,$$

the nonnegativity condition

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^* x^* = 0.$$

d. The complementary slackness condition requires that either $x^* = 0$ or $\lambda^* = 0$, but the first-order condition implies that if $x^* = 0$ then $\lambda^* = 0$ too and vice-versa. Hence, $x^* = 0$ and $\lambda^* = 0$. These solutions confirm the guess made in part (a), but also reveal that this is a case in which the constraint holds as an equality but does not bind, in the sense that the multiplier is zero at the optimum.

3. The Constraint Qualification

A consumer chooses consumption c of a single good in order to maximize the utility function $U(c)$ subject to the budget constraint $I \geq pc$, where $I > 0$ is the consumer's income and $p > 0$ is the price of the good. Suppose that the utility function is strictly increasing, so that $U'(c) > 0$ for all values of c.

a. Since the utility function is strictly increasing, it will always be optimal for the consumer to spend all of his or her income on the single good. Hence, $c^* = I/p$.

b. With the Lagrangian defined as

$$L(c, \lambda) = U(c) + \lambda(I - pc),$$

the Kuhn-Tucker conditions are

$$L_1(c^*, \lambda^*) = U'(c^*) - \lambda^* p = 0,$$

$$L_2(c^*, \lambda^*) = I - pc^* \geq 0,$$

$$\lambda^* \geq 0,$$

and

$$\lambda^*(I - pc^*) = 0.$$

Since $U'(c) > 0$ for all values of c, the first-order condition requires $\lambda^* > 0$, which then implies via the complementary slackness condition that $c^* = I/p$. Substituting this result back into the first-order condition yields the solution $\lambda^* = U'(I/P)$.

c. With the Lagrangian defined as

$$L(c, \lambda) = U(c) + \lambda(I - pc)^3,$$

the Kuhn-Tucker conditions are

$$L_1(c^*, \lambda^*) = U'(c^*) - 3p\lambda^*(I - pc^*)^2 = 0,$$

$$L_2(c^*, \lambda^*) = (I - pc^*)^3 \geq 0,$$

$$\lambda^* \geq 0,$$

and

$$\lambda^*(I - pc^*)^3 = 0.$$

But substituting the value of $c^* = I/p$ found above into the new first-order condition yields

$$U'(I/P) - 3p\lambda^*(I - I)^2 = U'(I/P) = 0,$$

which cannot hold if $U'(c) > 0$ for all values of c. What has gone wrong is that with the constraint written in an unnecessarily complicated way, the constraint qualification, which requires that

$$3(I - pc^*)^2 \neq 0,$$

fails to hold.

4. Perfect Substitutes

Suppose a consumer likes two goods, good 1 and good 2, which he or she views as perfect substitutes: perhaps they are just different brands of the same basic product. A utility function that captures this idea takes the linear form

$$U(c_1, c_2) = c_1 + c_2.$$

Let I denote the consumer's income and let p_1 and p_2 denote the prices of the two goods. In this case, the linearity of the utility function means that nonnegativity conditions for c_1 and c_2 need to be imposed for the problem to make economic sense. Accordingly, associated with the consumer's problem

$$\max_{c_1, c_2} c_1 + c_2 \text{ subject to } I \geq p_1 c_1 + p_2 c_2,\ c_1 \geq 0,\ \text{and } c_2 \geq 0,$$

define the Lagrangian

$$L(c_1, c_2, \lambda, \mu_1, \mu_2) = c_1 + c_2 + \lambda(I - p_1 c_1 - p_2 c_2) + \mu_1 c_1 + \mu_2 c_2.$$

a. The solution is for the consumer to spend all of his or her income on the good with the lowest price: $c_1^* = I/p_1$ and $c_2^* = 0$ if $p_1 < p_2$ and $c_1^* = 0$ and $c_2^* = I/p_2$ if $p_1 > p_2$. If $p_1 = p_2$, then any values of c_1^* and c_2^* satisfying the budget constraint with equality will be optimal.

b. According to the Kuhn-Tucker theorem, the values of c_1^* and c_2^* that solve the consumer's problem, together with the associated values of λ^*, μ_1^*, and μ_2^*, must satisfy the first-order conditions

$$L_1(c_1^*, c_2^*, \lambda^*, \mu_1^*, \mu_2^*) = 1 - \lambda^* p_1 + \mu_1^* = 0$$

and

$$L_2(c_1^*, c_2^*, \lambda^*, \mu_1^*, \mu_2^*) = 1 - \lambda^* p_2 + \mu_2^* = 0,$$

the constraints

$$L_3(c_1^*, c_2^*, \lambda^*, \mu_1^*, \mu_2^*) = I - p_1 c_1^* - p_2 c_2^* \geq 0,$$

$$L_4(c_1^*, c_2^*, \lambda^*, \mu_1^*, \mu_2^*) = c_1^* \geq 0,$$

and

$$L_5(c_1^*, c_2^*, \lambda^*, \mu_1^*, \mu_2^*) = c_2^* \geq 0,$$

the nonnegativity conditions

$$\lambda^* \geq 0,$$

$$\mu_1^* \geq 0,$$

and

$$\mu_2^* \geq 0,$$

and the complementary slackness conditions

$$\lambda^*(I - p_1 c_1^* - p_2 c_2^*) = 0,$$

$$\mu_1^* c_1^* = 0,$$

and

$$\mu_2^* c_2^* = 0.$$

c. The first-order conditions and the nonnegativity conditions for μ_1^* and μ_2^* imply that $\lambda^* > 0$, so that the budget constraint must always bind. Hence, there are three possibilities to consider. The first possibility is that $\mu_1^* > 0$ and $\mu_2^* = 0$. In this case, the complementary slackness condition requires that $c_1^* = 0$ and the binding budget constraint implies that $c_2^* = I/p_2$. The first-order conditions require that $\lambda^* = 1/p_2$ and $\mu_1^* = \lambda^* p_1 - 1 = p_1/p_2 - 1$. But this last condition is consistent with $\mu_1^* > 0$ only if $p_1 > p_2$. The second possibility is that $\mu_1^* = 0$ and $\mu_2^* > 0$. In this case, reasoning analogous to that above implies that $c_1^* = I/p_1$, $c_2^* = 0$, $\lambda^* = 1/p_1$, and $\mu_2^* = p_2/p_1 - 1$. But this last condition is consistent with $\mu_2^* > 0$ only if $p_2 > p_1$. The final possibility is that $\mu_1^* = \mu_2^* = 0$. The first-order conditions imply that this case can only occur when $p_1 = p_2$, so that $\lambda^* = 1/p_1 = 1/p_2$ and any values of c_1^* and c_2^* satisfying the budget constraint with equality will work. These solutions serve to confirm the guesses from part (a) above.

5. Elasticities of Demand

Consider a consumer who purchases three goods to maximize utility subject to a budget constraint. Assuming that the utility function is such that nonnegativity constraints on the choice variables can be ignored, and extending the notation used above in the obvious way, the consumer's problem can be written as

$$\max_{c_1,c_2,c_3} U(c_1,c_2,c_3) \text{ subject to } I \geq p_1c_1 + p_2c_2 + p_3c_3.$$

Suppose that the utility function is also such that is it possible to find functions $c_1^*(p_1,p_2,p_3,I)$, $c_2^*(p_1,p_2,p_3,I)$, and $c_3^*(p_1,p_2,p_3,I)$ that uniquely determine the optimal choices in terms of the parameters measuring prices and income (note that here, the subscripts refer to the three goods and not to the derivatives of the functions). Under most circumstances, these functions, which correspond to "Marshallian demand curves" for each of the three goods, can be expected to satisfy two basic conditions. First, under the assumption that the budget constraint binds at the optimum, it must be that

$$p_1c_1^*(p_1,p_2,p_3,I) + p_2c_2^*(p_1,p_2,p_3,I) + p_3c_3^*(p_1,p_2,p_3,I) = I \tag{1}$$

for all values of p_1, p_2, p_3, and I. Second, because increasing or decreasing all three prices and the consumer's income by the same proportions has no effect on the consumer's optimal choices, it must be that

$$c_i^*(rp_1,rp_2,rp_3,rI) = c_i^*(p_1,p_2,p_3,I) \tag{2}$$

for any value of $r > 0$ for each $i = 1,2,3$; that is, the Marshallian demands are "homogeneous of degree zero."

Now, for each $i = 1,2,3$ and $j = 1,2,3$, let

$$\varepsilon_{i,j} = \frac{p_j}{c_i^*(p_1,p_2,p_3,I)}\frac{\partial c_i^*(p_1,p_2,p_3,I)}{\partial p_j}$$

denote the elasticity of demand for good i with respect to the price of good j, and for each $i = 1,2,3$, let

$$\eta_i = \frac{I}{c_i^*(p_1,p_2,p_3,I)}\frac{\partial c_i^*(p_1,p_2,p_3,I)}{\partial I}$$

denote the income elasticity of demand for good i. Finally, for each $i = 1,2,3$, let

$$s_i = \frac{p_ic_i^*(p_1,p_2,p_3,I)}{I}$$

denote the share of his or her total income that the consumer spends on good i.

a. Differentiate both sides of (1) with respect to one of the prices p_j to obtain

$$p_1\frac{\partial c_1^*(p_1,p_2,p_3,I)}{\partial p_j} + p_2\frac{\partial c_2^*(p_1,p_2,p_3,I)}{\partial p_j} + p_3\frac{\partial c_3^*(p_1,p_2,p_3,I)}{\partial p_j} + c_j^*(p_1,p_2,p_3,I) = 0.$$

Now use the definition of the price elasticities $\varepsilon_{i,j}$ to rewrite this expression as

$$\begin{aligned} p_1[c_1^*(p_1,p_2,p_3,I)/p_j]\varepsilon_{1,j} + p_2[c_2^*(p_1,p_2,p_3,I)/p_j]\varepsilon_{2,j} & \\ +p_3[c_3^*(p_1,p_2,p_3,I)/p_j]\varepsilon_{3,j} + c_j^*(p_1,p_2,p_3,I) &= 0. \end{aligned}$$

Finally, multiply through by p_j, divide through by I, and rearrange to obtain

$$s_1\varepsilon_{1,j} + s_2\varepsilon_{2,j} + s_3\varepsilon_{3,j} = -s_j.$$

This relationship is often referred to as a statement of "Cournot aggregation." It implies, for instance, that if good i is a Giffen good, with $\varepsilon_{j,j} > 0$, then at least one of the other goods must be strongly complementary, in the sense that $\varepsilon_{i,j} < 0$ must be large in absolute value.

b. Differentiate both sides of (1) with respect to I to obtain

$$p_1\frac{\partial c_1^*(p_1,p_2,p_3,I)}{\partial I} + p_2\frac{\partial c_2^*(p_1,p_2,p_3,I)}{\partial I} + p_3\frac{\partial c_3^*(p_1,p_2,p_3,I)}{\partial I} = 1.$$

Now use the definition of the income elasticities η_i to rewrite this expression as

$$p_1[c_1^*(p_1,p_2,p_3,I)/I]\eta_1 + p_2[c_2^*(p_1,p_2,p_3,I)/I]\eta_2 + p_3[c_3^*(p_1,p_2,p_3,I)/I]\eta_3 = 1$$

or, more simply,

$$s_1\eta_1 + s_2\eta_2 + s_3\eta_3 = 1.$$

This relationship is often referred to as a statement of "Engel aggregation." It implies that: (a) not every good can be inferior, with $\eta_i < 0$, (b) not every good can be a luxury, with $\eta_i > 1$, and (c) if all goods have the same income elasticity of demand, then that common income elasticity must equal one.

c. Differentiate both sides of (2) with respect to r to obtain

$$\begin{aligned} p_1\frac{\partial c_i^*(rp_1,rp_2,rp_3,rI)}{\partial p_1} + p_2\frac{\partial c_i^*(rp_1,rp_2,rp_3,rI)}{\partial p_2} & \\ +p_3\frac{\partial c_i^*(rp_1,rp_2,rp_3,rI)}{\partial p_3} + I\frac{\partial c_i^*(rp_1,rp_2,rp_3,rI)}{\partial p_I} &= 0. \end{aligned}$$

Now divide through by $c_i^*(rp_1,rp_2,rp_3,rI)$ and set $r=1$ to obtain

$$\varepsilon_{i,1} + \varepsilon_{i,2} + \varepsilon_{i,3} + \eta_i = 0.$$

This relationship states in elasticity form the implications of Euler's theorem on homogenous functions: see Simon and Blume, p.491, for details.